Structures in Science and Metaphysics

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ABSTRACT. In this paper, we develop a formalization of a metaphysics of structures and provide new applications to geometry and to metaphysics. We adopt the theory of set-theoretical structures formulated in da Costa and French [15] and in da Costa and Rodrigues [16]. The applications are made in the context of extending the proposal advanced in Caulton and Butterfield [11] to a higher-order metaphysics.

Introduction

This paper has three main purposes: (1) It aims to discuss some aspects of the general theory of set-theoretic structures as presented in da Costa and French [15]. However, our exposition is in principle independent of the contents of this book, being in good part based on the paper da Costa and Rodrigues [16]. The theory of structures studied here can be employed in the axiomatization of scientific theories, particularly in physics. The theory is related to that of Bourbaki [4], but our emphasis is on the semantic dimension of the subject, which is not in agreement with Bourbaki’s syntactic views. Set-theoretic structures are important, among other things, because they constitute the basis for a philosophy of science, as it was delineated in da Costa and French [15] (see also Suppes [34]). After developing the framework, we examine some aspects of geometry in order to make clear how the central notions of the theory can be applied to higher-order structures found in science. The case of geometry is significant, since pure geometric structures can be considered not only as abstract mathematical constructs, but also as essential tools for the formulation of physical theories, in particular the theories of space and time.

(2) The paper also aims to discuss how to extend ideas of Caulton and Butterfield [11] to what may be called higher-order metaphysics. In this metaphysics, in addition to first-order objects, we also find their corresponding properties and relations of any type in a convenient type hierarchy.

(3) Finally, the paper outlines the first steps of the formalization of a metaphysics of structures, or structural metaphysics, inspired by ideas of authors such as French and Ladyman (see, for example, Ladyman [29], French and Ladyman [19], and French [18]). This kind of metaphysics may be seen as an adaptation of the higher-order metaphysics of structures implicit in Caulton and Butterfield [11].

1 Set theory, structures, and languages

In this section we present the set-theoretic framework that will be applied subsequently in the paper, which summarizes some concepts and results of da
Costa and Rodrigues [16] and da Costa and Bueno [13]. All our (set-theoretic) constructions are implemented in Zermelo-Fraenkel set theory, which is the basis for the framework (in particular, we take languages to be a certain kind of free algebra).

The set $T$ of types is defined as follows:

1. The symbol $i$ belongs to $T$;
2. If $t_0, t_1, \ldots, t_{n-1} \in T$, then $\langle t_0, t_1, \ldots, t_{n-1} \rangle$, $1 \leq n < \omega$ also belongs to $T$.
3. The elements of $T$ are only those given by clauses 1 and 2.

The order of a type $t$, $\text{ord}(t)$ is introduced as follows:

1. $\text{ord}(i) = 0$.
2. $\text{ord}(\langle t_0, t_1, \ldots, t_{n-1} \rangle) = \max\{\text{ord}(t_0), \text{ord}(t_1), \ldots, \text{ord}(t_{n-1})\} + 1$

The transitive closure $[t]$ of a type $t$ is given by the clauses:

1. $t \in [t]$.
2. If $t = \langle t_0, t_1, \ldots, t_{n-1} \rangle$, then $t_0, t_1, \ldots, t_{n-1}$ belong to $[t]$.
3. The elements of $[t]$ are only those given by clauses 1 and 2.

We have: $[t] = [t_0] \cup [t_1] \cup \ldots \cup [t_{n-1}]$. All elements of $[t]$, different from $t$, have orders strictly less than the order of $t$. On the other hand, $[i] = \emptyset$ and $\text{ord}(i) = 0$. There is a partial ordering in the set of types, $\leq$, defined by the condition: $t_1 \leq t_2$ if $t_1 \in [t_2]$. Any decreasing sequence of types is finite, i.e., the set of types is regular.

If $D$ is a set, then we define a function $\tau_D$ or, to simplify, $\tau$, whose domain is $T$, by the following clauses:

1. $\tau(i) = D$.
2. If $t_0, t_1, \ldots, t_{n-1} \in T$, then $\tau(\langle t_0, t_1, \ldots, t_{n-1} \rangle) = \mathcal{P}(\tau(t_0) \times \tau(t_1) \times \ldots \times \tau(t_{n-1}))$, where $\mathcal{P}$ and $\times$ are the symbols for the power-set and the Cartesian product, respectively.

The set $\bigcup \text{range}(\tau_D)$ is denoted by $\varepsilon(D)$, and is called the scale based on $D$. The objects of $\tau(i)$ are called individuals of $\varepsilon(D)$, and the objects of $\tau(t)$, $\text{ord}(t) > 0$, are called objects or relations of type $t$; the type of individuals is $i$. We also have: $\varepsilon(D) = \bigcup(\text{range}(\tau_D) - \{D\})$.

The cardinal $k(D)$, defined by the condition

$$k(D) = \sup\{D, \mathcal{P}(D), \mathcal{P}(\mathcal{P}(D)), \ldots\},$$

is the cardinal associated with $\varepsilon(D)$. (Sometimes, instead of $k(D)$, we write $k_D$.)
A sequence is a function whose domain is an ordinal number, finite or infinite. $\hat{b}_\lambda$ is the range of the sequence $b$, and $\lambda$ is an ordinal.

We call structure $e$ with basic set $D$ an ordered pair

$$e = \langle D, r_\iota \rangle$$

where $r_\iota$ is a sequence of elements of $\varepsilon(D)$. We also call $D$ the domain of $e$, and it is supposed to be non-empty. $D$ and the terms of $r_\iota$ are the primitives elements of $e$; the elements of $D$ are the individuals of $e$. We shall identify such individuals with their unit sets when there is no danger of confusion. $\varepsilon(D)$ and $k(D)$ are the scale and the cardinal associated to $e$, respectively. In all cases, the ordinal which is the domain of $r_\iota$ is strictly less than $k(D)$. $\hat{\varepsilon}(D)$ is the strict scale associated to $e$, and contains all relations of $\varepsilon(D)$, including the unary relations, that is, sets.

The order of an object of $\varepsilon(D)$ is the order of its type. The order of $e = \langle D, r_\iota \rangle$, denoted by $\text{ord}(e)$, is defined as follows: if there is the greatest order of the objects of the range of $r_\iota$, then $\text{ord}(e)$ is this greatest order; otherwise, $\text{ord}(e) = \omega$.

$L_{\omega_k}^\omega(R)$ is the higher-order, infinitary language introduced in da Costa and Rodrigues [16]. Its blocks of quantifiers are finite, and conjunctions and disjunctions have length strictly less than $k(D)$, when $k(D) > \omega$; if $k(D) = \omega$, such conjunctions and disjunctions are always finite. $R$ is the set of constants of the language, each having a fixed type.

$L_{\omega_k}^\omega(R)$ can be interpreted in structures of form

$$e = \langle D, r_\iota \rangle,$$

the constants denoting $D$ and the objects $r_\iota$, each constant possessing the same type as that of the object it denotes. The sequence $r_\iota$ has as its domain an ordinal strictly less than $k(D)$, usually finite or denumerable. In what follows we may identify a constant with the object it denotes; this will be done when there is no danger of confusion. In the structure $e = \langle D, r_\iota \rangle$, $D$ and $r_\iota$ are called the primitive concepts of $e$.

Sentences of our languages are formulas without free variables. We say that a sentence $\xi$ of a language, interpreted in a structure

$$e = \langle D, r_\iota \rangle$$

is true in this structure in analogy with the case of first-order (finitary) logic. To express that $\xi$ is true in $e$ we write:

$$e \models \xi.$$

Notions such as model, definability, etc. are defined as in da Costa and Rodrigues [16].

From now on, we usually suppose that in the structure $e = \langle D, r_\iota \rangle$ the family $r_\iota$ of relations is finite (i.e., the domain of $r_\iota$ is a finite ordinal). When $r_\iota$ is infinite this will be explicitly stated. As a consequence, every structure will have, usually, an order that is a positive integer or zero.
To each structure of order \( n \), \( e = \langle D, r_\iota \rangle \), it is associated a finitary language of order \( n \) \( L^n_{\omega}(R) \) and an infinitary language \( L^n_{\omega_k D}(R) \), or to simplify \( L^n_{\omega_k}(R) \), where \( R \) is the range of \( r_\iota \) and whose variables have types of order at most \( n \). These languages are the strict languages associated with the structure \( e \) of order \( n \).

The kind of structure \( e = \langle D, r_\iota \rangle \) is the sequence \( t_\iota \), where \( t_\iota \) is the type of \( r_\iota \). The two languages associated with a given structure are both interpretable in this structure. But, for convenience, isomorphic structures of the same kind will be identified, despite the obvious differences between them.

In the languages associated with a structure \( e = \langle D, r_\iota \rangle \) of order \( n \), we may add variables of any order \( m \), with \( m > n \); in particular, \( m \) may be \( \omega \). Clearly such languages are interpretable in \( e = \langle D, r_\iota \rangle \), via \( \varepsilon(D) \). In this case, \( e \) becomes the basis of a semantics of those languages. In particular, Peano arithmetic is a first-order structure, but there are corresponding arithmetic theories of any order whatsoever.

If \( e = \langle D, r_\iota \rangle \) is a structure of order \( n \), \( e^{(m)} = \langle D, s_\lambda \rangle \) will be the structure in which \( s_\lambda \) is the sequence of all relations of \( \varepsilon(D) \) of order \( \leq m \) definable in the wide sense in \( e = \langle D, r_\iota \rangle \) by the means of the infinitary language associated with \( e \). Sometimes, all the relations \( s_\lambda \) are definable in the wide sense in terms of a finite number of relations in the range of \( s_\lambda \). Thus, it is possible to take \( s_\lambda \) as a finite sequence. \( e^{(m)} \) constitutes the \( m \)-order counterpart of \( e \), and the \( m \)-order semantics of \( e \) is the semantics of \( e^{(m)} \). For example, although elementary Euclidean geometry is not a first-order structure, it is possible to investigate its first-order counterpart (see Tarski [35]).

Despite the fact that the language of set theory is first-order, structures of any order whatsoever can be studied in it and the corresponding theory of sets. This is, of course, a different situation from the case of elementary Euclidean geometry just mentioned, which involves the study of the first-order counterpart of a non-first-order structure. The semantic interplay between a language (that is, a free algebra) and a structure can be treated by means of set theory (say, Zermelo-Fraenkel set theory) even when the structure is of order \( n > 1 \).

The theory of the kind of structure of \( e = \langle D, r_\iota \rangle \), or the species of structures to which \( e \) belongs, can be formulated in one of its associated languages, sometimes enlarged by extra variables and constants of higher-order types, or in the language of set theory. This is accomplished by the choice of appropriate postulates, formulated in the chosen language.

Theories, species of structures or axiomatic systems are formulated as follows: (1) It is given a language, which may be that of set theory extended by the addition of new primitive terms, or associated with structures of a certain kind. (2) The postulates and rules of the underlying logic are those naturally connected with the language of clause (1) (in the case of set theory, employed as the underlying framework, its axioms are clearly included among the postulates of the theory). (3) Specific postulates for the theory are also introduced. (4) By means of the logic of clause (2) and the specific axioms, the concept of theorem is defined.

In this way, one immediately obtains the common syntactic and semantic
results regarding theories. Our notion of theory, axiomatic system or species of structures is, therefore, more general than the one articulated in Bourbaki [4], since we are not restricted to syntactic considerations alone.

Two important concepts are those of transportable formula and of intrinsic term that we will introduce informally. The postulates of a theory are supposed to be transportable formulas. A formula of the language of theory $T$, interpretable in structures of the kind of the structure $e$, is transportable if, and only if, being true in one structure of this kind it is also true in every structure isomorphic to it.

Terms may be joined to the language of a theory as new constants or with the help of the description operator. We say that term $t$ is intrinsic in a theory $T$ if, given any two isomorphic models $e$ and $e'$ of $T$, then for any isomorphism $\xi$ between $e$ and $e'$, the denotation of $t$ in $e'$ is $\xi(a)$, where $a$ is the denotation of $t$ in $e$.\footnote{The notions of transportable formula and of intrinsic term were introduced by Bourbaki [4] (compare with da Costa and Rodrigues [16], and da Costa and Chuaqui [14]).}

These considerations can be extended to the case in which the starting point is a finite sequence $D_1, D_2, \ldots, D_m$ of basic sets and the relations $r_i$ involve elements of the scale $\varepsilon(D_1 \cup D_2 \cup \ldots \cup D_m)$. The notions of type, order of a type, scale, individuals, etc. are easily adapted to this new case. A structure, then, is a set-theoretic construct

$$e = \langle D_1, D_2, \ldots, D_m, r_i \rangle,$$

where $D_1, D_2, \ldots, D_m$ is the finite sequence of basic sets and $r_i$ are the primitive relations of $e$. The sequence $D_1, D_2, \ldots, D_m$ (or, to abbreviate, $D_\delta$) is composed by arbitrary (non-empty) sets, and the notions of language associated with $e$, theory or species of structures of the kind of $e$, etc. are introduced without difficulty. We usually say that the sets $D_\delta$ are endowed with the species of structures of $e$.

In most cases, some or all basic sets are supposed to be endowed with previously defined structures (for example, in the species of real vector spaces, the scalars are the real numbers); these basic sets are called auxiliary sets. The other basic sets constitute the strict basic sets. Normally, structures have at least one strict basic set.

If the family $r_i$ in the structure $e = \langle D, r_i \rangle$ is empty, then $e$ reduces to the bare set $D$.

An isomorphism between two structures of the same kind $e_1 = \langle D_\delta, r_i \rangle$ and $e_2 = \langle B_\delta, s_i \rangle$ is a family of bijections between $D_\delta$ and $B_\delta$ satisfying obvious conditions, plus the following: the bijection between two corresponding auxiliary sets is, for simplicity, the relation of identity. In other words, isomorphisms must keep invariant the corresponding auxiliary sets and their associated structures.

Therefore, the structure

$$e = \langle D_1, D_2, \ldots, D_m, r_i \rangle$$

may be written as follows:

$$e = \langle D_\delta, A_\delta, r_i \rangle.$$
where $D_δ$ is the family of strictly basic sets and $A_δ$ is the family of auxiliary sets.

Structures are commonly presented as finite sequences of previously defined structures arranged by appropriate relations among them. So, the notion of species of structures (theory or axiomatic system) has to be adjusted to the new situation. But that is not difficult to be done from the conceptual point of view.

For instance, a Riemannian geometric structure (Riemannian space or Riemannian geometry) is an ordered pair

$$R = \langle D, m \rangle,$$

where $D$ is a (real) differential manifold and $m$ is a Riemannian metric on $D$ (that is, a symmetric, positively defined, 2-covariant tensor field on $D$). The theory of Riemannian spaces has as its models structures such as $R = \langle D, m \rangle$.

Another sort of structures, usually occurring in physics, is that of a bundle. A bundle is a triplet

$$F = \langle E, \pi, D \rangle$$

where $E$ (the bundle space) and $D$ (the base space) are topological spaces, and $\pi$ (the bundle projection) a continuous map of $E$ on $D$.

Structures presented as sequences of other structures are reducible to structures in the standard sense, although it must be clear what are the basic sets and the auxiliary sets. The reduction is expressed, among other things, by the description of the inter-relations imposed on the component structures. For example, in the case of bundles, the reduction to the standard version may be as follows:

$$F = \langle A, B, t_A, t_B, s \rangle$$

where $t_A$ and $t_B$ are collections of subsets of $A$ and $B$, respectively, and $s$ is a function of $A$ on $B$. We then introduce the postulates of $F$, which are the following ones:

1. $t_A$ is a topology on $A$;
2. $t_B$ is a topology on $B$;
3. $s$ is a continuous function of the topological space $\langle A, t_A \rangle$ on the topological space $\langle B, t_B \rangle$.

The reduction of a Riemannian structure to its standard formulation would be more complex, but it can still be formulated.

2 Definability and expressibility

In some cases, to simplify the exposition, we identify relations with their names. In the structure $e = \langle D, r_\iota \rangle$, we say that $s \in \varepsilon(D)$ is strictly definable in $e$ if the following condition is met: there is a formula $F(x)$ of the finitary language associated with $e$, in which $x$ is its sole free variable, such that:

$$e' \models \forall x (x = s \leftrightarrow F(x))$$
in the language just mentioned and to which the new primitive term $s$ was added, where $e'$ is the structure $e$ with the relation $s$ included.

Let $e = \langle D, r_i \rangle$ be a structure, $\varepsilon(D)$ its scale, $s$ an element of $\varepsilon(D)$, and $l_\mu(\mu < kD)$ a sequence of objects of $e = \langle D, r_i \rangle$. We then define, by induction, that the expression $s$ is expressible in the sequence $l_\mu$ in structure $e'$ as follows:

1. if $s$ is definable in the strict sense in the structure $e$ to which we have joined the terms of $l_\mu$ as new primitive relations, then $s$ is expressible in the sequence $l_\mu$ in $e'$;

2. if $s$ is not an individual of $e$, then $s$ is expressible in $l_\mu$ in $e'$ if every element of $s$ is expressible in $l_\mu$ in $e$;

3. if $s$ is expressible in the sequence $b_\alpha$ of elements of $\varepsilon(D)$ in the structure $e$ and every element of $b_\alpha$ is expressible in $l_\mu$ in $e$, then $s$ is expressible in $l_\mu$ in the structure $e$.

Expressibility is equivalent to definability in the infinitary language associated with $e$ extended by the addition of the term $s$ (see da Costa and Rodrigues [16]). Instead of ‘expressibility’, we also use the expression ‘definability in the wide sense’.

A definition of the new constant $c$ joined to the language of the theory $T$ is an expression of the form $c = \iota x F(x)$, such that:

$$T \vdash \exists x F(x),$$

where $F(x)$ is a formula of the original language of $T$ (without $c$) with $x$ as its only free variable.

The previous definition can be adjusted to the cases in which the constant $c$ belongs to the language of $T$, or in which $T$ has no specific postulates (such postulates constitute the empty set), or when the notion of semantic consequence ($\models$) is used instead of that of syntactic consequence.

Let $L$ be a language, $e_1 = \langle D, r_i \rangle$ be a structure whose language $L(D, r_i)$ is an extension of $L$ by the introduction of the new constant $D$ and the family $r_i$ of new primitive constants, and $e_2 = \langle D, s_\lambda \rangle$ be another structure having its language defined analogously. In this case, $e_1$ is said to be strictly (widely) equivalent to $e_2$ if every $s_\lambda$ is definable in the strict (wide) sense in terms of $D$ and $r_i$, and conversely. For example, in the language of set theory, two topological structures, $t_1 = \langle D, v \rangle$ and $t_2 = \langle D, A \rangle$, characterized by the family of neighborhoods and the set of open sets, respectively, are strictly equivalent. In the same language, the structure of natural numbers, in the sense of von Neumann, is widely equivalent to the structure of real numbers, conceived as Cauchy sequences of rational numbers; however, there are real numbers that are not strictly definable in terms of natural numbers, although they are widely definable in terms of these numbers.

$L$ is the language of set theory. We assume that the postulates of $L$ are the common postulates of $ZF$ (Zermelo-Fraenkel set theory). Suppose that $T_1$ is a theory whose language is $L(D, r_i)$, where $D$ denotes a set and $r_i$ is a family of relations over $D$. We denote by $T_2$ another theory whose language is $(LE, s_\lambda)$, similarly defined. Under these conditions, $T_1$ and $T_2$ are equivalent if:
1. every $s_\lambda$ and $E$ are definable by formulas of $L(D,r_\lambda)$ in $T_1$, and every $r_\iota$ and $D$ are definable in $L(E,s_\lambda)$ by formulas of $L(E,s_\lambda)$ in $T_2$;

2. every postulate of $T_2$, reformulated in the language of $T_1$, is provable in $T_1$, and conversely.\(^2\)

The usual metamathematical results involving mathematical structures and species of structures can be obtained in this approach. For instance, we have:

1. Any two models of second-order arithmetic are isomorphic. (2) First-order arithmetic is not categorical. (3) In first-order languages, as usually formulated (with a first-order semantics), the notion of finite group is not axiomatizable. (4) In a convenient infinitary, first-order arithmetic, all subsets of natural numbers are widely definable. (5) In set theory, any structure of any order can be extended to a rigid structure, by adding a finite number of new relations (da Costa and Rodrigues [16]).

3 Theories and operations with structures

As noted, structures are correlated with, at least, three classes of language: the two classes associated with them and that of set theory. The orders of the languages associated with a structure $e$, languages that can be finitary or infinitary, are the same as that of $e$. Nonetheless, as also noted, any structure can be studied in set theory, i.e., with the help of the tools of set theory, whose language is first order. We need to be careful here, since a given structure of order $n$ has an intended semantics of order $n$. In a certain sense, the order of structure $e$ is kept invariant in set theory, but the membership relation strongly enriches the intended semantics of $e$. Concepts like definability, definability in the wide sense, discernibility, indiscernibility, individuality, etc., to be examined below, are essentially language dependent.

From now on, we reserve the expression ‘species of structures’ for theories axiomatized in set theory (ZF with its axioms), but whose language is extended by adding extra symbols denoting relevant sets and relations. When a different language is employed, this will be made explicit.

Species of structures have important features. We highlight some below:

1. When a species of structures is based on various sets, it is easy to transform the species into an equivalent one with only one basic set. However, if the underlying language of a theory differs from that of set theory, for example, if it is a first-order language involving various specific primitive symbols (constants), this transformation is not always possible. To implement the transformation, we obviously need the resources of set theory. Shoenfield [33], for instance, axiomatizes what is essentially second-order arithmetic in a two-sorted, first-order language. But it is not possible to reduce the basic sets to only one in the language used. Some non-standard models are also included via the corresponding first-order semantics.

\(^2\)There are some obvious variations in the definition of theory equivalence. But we need not discuss them here (see, for instance, Shoenfield [33] for the case of finitary, first-order languages).
2. The usual frameworks for the study of higher-order structures are set theory or higher-order logic. In both cases, the intended, standard semantics is adopted. However, a many-sorted, first-order logic can also be employed, with the inclusion of first-order structures.

3. The specific axioms of a species of structures are always supposed to be transportable and its terms intrinsic. When the underlying logic of a theory is type theory (higher-order logic) these conditions are automatically satisfied (see da Costa and Chuaqui [14]).

4. Starting with initial structures, new structures can be obtained in set theory via various methods. The following are worth mentioning: axiomatization, (structural) deduction, combination, structural derivation, and the use of universal mappings (see Bourbaki [4]). We consider each of them in turn.

*Axiomatization* is the general way to introduce new species of structures. This is done, as indicated above, in accordance with the definition of species of structures. When other languages are used, the process is similar, but with suitable adjustments.

As an illustration, here are some examples of species of structures: (i) a *groupoid* is a set endowed with a binary operation; (ii) a *semigroup* is a groupoid whose operation is associative; (iii) a *semigroup* with identity is a manoid. Moreover, species of structures such as topological space, lattice, Boolean algebra, loop and topological group can also be defined.

*Deduction of species of structures* (or *structural deduction*) consists in a method of formulation of new structures that is best illustrated by an example (for a detailed description, see Bourbaki [4]). The real projective plane depends on the species of real numbers:

\[ \mathcal{R} = \langle R, +, \times, 0, 1, \leq \rangle, \]

which is a higher-order structure. As is well known, it can be characterized as follows: (1) The starting point is a set of triples of real numbers. (2) Projective points are then introduced as classes of equivalence of triples. (3) Finally, primitive concepts of the real projective plane are introduced: the set of points \( P \), the set of lines \( L \), and the relation of incidence \( I \) (if a point and a line are incident, the point lies on the line or the line passes through the point). The structure of a real projection plane is, thus, a triplet:

\[ P = \langle P, L, I \rangle \]

satisfying proper axioms. In other words, the real projective plane was obtained, by deduction, from \( \mathcal{R} \) through the intrinsic terms \( P, L \) and \( I \) (whose intrinsic nature can be easily verified). It is clear that the notation:

\[ \langle P, L, I \rangle \]

for the real projective plane is no more than an abbreviation, omitting many details; nonetheless, it is a useful notation.
Numerous species of structures are obtained via combination of other species. For example, a topological group $G$ involves the species of structures of group and that of topological space:

$$G = \langle A, B \rangle,$$

where $A$ is a group and $B$ a topological space, the operations of $A$ being continuous relatively to the topology of $B$. Various species of structures are obtained in this way, such as differentiable manifold, analytic manifold, and Grassmann algebra.

The operation of structural derivation (of structures) is related to the notion of morphism, as discussed in Bourbaki [4]. Examples of structures of this sort are the inverse image structure, induced structure, product structure, direct image structure, and quotient structure.

Free structures are introduced with the help of universal mappings. For example, mathematically speaking, a formal language is a kind of free algebraic structure.\(^3\)

4 The Erlangen Program and some of its extensions

Klein’s Erlangen Program is well known in geometry. Veblen, for instance, summarizes it as follows:

The system of definitions and theorems which expresses properties invariant under a given group of transformations may be called, in agreement with the point of view expounded in Klein’s Erlangen Program, a geometry. (Veblen and Young [36], volume I, p. 71.)

Reflecting on the concept of space, Wilder notes:

In modern mathematics, the term ‘space’ has extremely broad connotations, the difference between ‘set’ and ‘space’ often being very slight, a ‘space’ being simply a set to which certain special properties have been added. The most common property is that having a metric, or distance function. (Wilder [38], p. 177.)

He then continues:

And, according to the point of view proposed by Klein in 1872, one may speak of the $\xi$-geometry of the space $S$ as the study of the properties of the space and its configurations that are invariant under [the group] $\xi$. That is, $\xi$-geometry of the space $S$ is the study of the $\xi$-geometry of $S$. (Wilder [38], p. 180.)

The concept of geometry à la Klein can, of course, be formulated in set theory (see Klein [24] and [25]). Let $D$ be a set and $g$ a group of transformations of $D$. A geometry in the sense of Klein is a structure $G = \langle \langle D, r_\xi \rangle g \rangle$, such that $g$ is the group of all transformations of $D$ which leave relations $r_\xi$ invariant, and

\(^3\)A natural way of formulating the concepts of species of structures and of structure is to use category theory (see Corry [10]). But it would take us too far afield to examine this issue here.
any invariant relation in $\varepsilon(D)$ is expressible in $\langle D, \varepsilon, r, i \rangle$. We call ‘figure’ any element of $\varepsilon(D)$ defined by a formula of the language of $G$, i.e., that of set theory plus $D$ and $r$, $i$. Two figures $A$ and $B$ are said to be equipollent if they are of the same type and there is a transformation of $g$ that transforms $A$ onto $B$. Equipollence is an equivalence relation on the set of figures of a fixed type. The invariants of figures are said to be invariants of $G$.

In most cases, $D$ is already endowed with some structure, and the relations of this structure are employed to define the relations $r$, $i$. If $g$ is the group of automorphisms of $e = \langle D, s, \lambda \rangle$, then $e' = \langle \langle D, s, \lambda \rangle, g \rangle$ is a Klein geometry. It is clear, then, that the concept of geometry underlies all mathematics. (An important generalization of Klein’s geometry is obtained by means of the notion of the action of a group on a structure, but here is not the place to pursue this issue.)

Another conception of geometry is that of Blumenthal and Menger [3], to which we now turn. Any figure $F$ of $G = \langle \langle D, r, i \rangle, g \rangle$ belongs to $\varepsilon(D)$ and so can be set-theoretically constructed starting with some subset $K$ of $D$. The structure $\langle K, r^*_i, F \rangle$, where $r^*_i$ is the restriction of $r$, $i$ to $K$, restriction easily definable, is called the structure associated with $F$. Given two figures $F_1$ and $F_2$, we can prove the following propositions:

PROPOSITION 1. If $F_1$ and $F_2$ are equipollent, then their associated structures are isomorphic (as substructures of $\langle D, r, i \rangle$). (We show below that de converse of this first proposition is not true.)

PROPOSITION 2. Any substructure of $\langle D, r, i \rangle$ is a figure.

Two figures are said to be $BM$-equipollent if their associated structures are isomorphic. Any $BM$-equipollence constitutes a relation of equivalence in the class of all figures of a given type. For Blumenthal and Menger, a geometry over a structure $\langle e, \approx \rangle$ may be such that $e$ is trivial, i.e., a bare set. Blumenthal and Menger note that:

This widens the applicability of the notion. We may, for example, consider the theory of cardinal numbers as a geometry over a set $\sum$, by defining two figures (subsets) of $\sum$ to be equivalent if and only if there exists a one-to-one correspondence between their elements. But in most of the important geometries, the convention that establishes the equivalence of figures makes use of the structure by virtue of which a set becomes the element-set of a space [structure]. Hence, one speaks more often of a geometry over a space than over a set. (Blumenthal and Menger [3], p. 28.)

Clearly, this remark is also true of $K$-geometry.
In Riemannian geometry and some of its extensions, due to the fact that the corresponding automorphism groups are commonly trivial, composed by the identity transformation only, Klein’s methodology isn’t enough to specify satisfactorily a geometry as a $K$-geometry. It is not difficult to realize that the inclusive view of Blumenthal and Menger cannot be profitably applied either. To address this problem, Veblen devised an interesting approach (see Veblen and Whitehead [37]).

According to Veblen, a pseudo-group is a non-empty set $S$ of bijective transformations of subsets of a fixed set, such that: (1) The inverse of any transformation of $S$ belongs to $S$. (2) If there is the product of two transformations of $S$, then it is in $S$. Obviously, every transformation group is a pseudo-group.

Let us now suppose, in what follows, that all functions satisfy standard conditions of continuity and differentiability. Veblen then defines, in a differential manifold $M$, a geometric object $k$ as follows: (a) $k$ is defined at every point $P$ of $M$; (b) $k$ determines, in every coordinate system $S$, at $P$, an ordered set of $m$ real numbers, called the components of $k$ in $S$ at $P$ ($m$ may be different from the dimension of $M$); (c) the components of $k$ in any coordinate system $S'$, at $P$, are functions of the components of $k$ in $S$ at $P$, the functions representing the coordinate transformations and the derivatives of these functions evaluated at $P$.

As a result, geometric objects are defined in every point of the manifold and are characterized by their transformation laws. They are, in fact, invariant under the pseudo-group of point transformations induced by local coordinate transformations. Examples of these objects are: (1) scalars, i.e., objects that have only one component, which is invariant under coordinate transformations, (2) vectors, and (3) tensors.

A differential geometry, in the sense of Veblen, is a triple $V = \langle H, g, k \rangle$, in which $M$ is a differential manifold, $g$ is a pseudo-group as above, and $k$ is a geometric object. When $k$ is a scalar, $V$ is essentially the manifold $M$. When $k$ is a (metric) tensor, $V$ is a Riemannian geometry. (See Kobayashi and Nomizu [26] for a discussion of differential geometry.)

The BM-metric geometry associated to a ‘space’ with a distance function defined between any two of its points, following Blumenthal and Menger, is the BM-geometry in which BM-equipollence is the relation of isomorphism between figures of a fixed type.

Here are some examples: (1) Geometries according to Klein: (1.1) the topology of $\mathcal{R}^3$ is the geometry whose group is the group the homeomorphisms of $\mathcal{R}^3$; (1.2) the ‘logic’ of a set is the $K$-geometry whose group of transformations is composed by all transformations of the set; (1.3) the affine geometry of $\mathcal{R}^3$, in the present context, corresponds to the group of affinities; (1.4) the Euclidean metric geometry is characterized by the group of isometries of $\mathcal{R}^3$. (2) $BM$-geometries: (2.1) the metric geometry of an infinitely dimensional Hilbert space; (2.2) the analogous metric geometry of a Minkowski space (the space of special relativity). (3) Veblen geometries ($V$-geometries): (3.1) any Riemannian geometry is a $V$-geometry; (3.2) differential manifolds are $V$-geometries, in which the geometric object is a scalar.
Geometries were envisaged above as set-theoretic structures. However, they can also be considered as species of structures or, more in line with current mathematical practice, as certain kind of theories. The axiomatic approach to geometry is firmly based on language, as is evident. In various cases, this language is that of set theory extended by new primitive terms (and postulates), but typed languages can also be employed, for instance those associated with the structures under study. Even when a particular geometry is formulated in a higher-order framework, it is possible, as we have already observed, to investigate its lower parts. This happens in the cases of the Euclidean real plane and of arithmetic (another example: we could investigate first-order versions of special relativity). (Axiomatic geometry, however, will not be taken into consideration here, since it is not relevant to the objectives of this paper.)

Let us return to structures, and let $t = \langle E, T \rangle$ be a topological space inside ZF, in which $E$ is the base space and $T$ its topology. The language in which we talk about $t$ is, thus, that of ZF plus the constants $t$, $E$ and $T$; the postulates are those of ZF plus the common topological axioms with some extra ones (such as those of Hausdorff). We denote by $G$ the group of homeomorphisms (the automorphisms) of $t$. Clearly any topological notion, that is, an element of the scale of $t$, definable in the structure in the strict sense or in the wide sense, is invariant under $G$. But an important point is also the converse question: Is any invariant notion definable in function of the primitive notions of $t$? In this connection, we have:

**Theorem 3.** In $t$, any invariant notion $d$ (that is, an element of the scale of $t$) is definable in the wide sense in the infinitary language associated with $t$ whenever $d$ is invariant under $G$.

**Proof.** To say that $d$ is definable in the wide sense in $t$ means that $d$ is expressible in the primitive constants of $t$. So, the theorem is a consequence of the results of da Costa and Rodrigues [16].

**Theorem 4.** In Euclidean metric geometry, a notion is definable in the wide sense (that is, expressible) if, and only if, it is invariant under the group of isometries.

**Proof.** See da Costa and Rodrigues [16].

In general, propositions analogous to the preceding ones are valid for all structures.

**Theorem 5.** There exist BM-geometries that aren’t $K$-geometries.

**Proof.** It suffices to observe that the BM-metric geometry of an infinite-dimensional Hilbert space is the study of isometries between figures of the space, and that there are isomorphisms between two figures that cannot be extended to an isometry of the whole space onto itself.

**Theorem 6.** There are structures in which some notions are expressible (definable in the wide sense) but not definable in the strict sense in the denumerable language associated with the structure.

**Proof.** In ZF the set of real numbers $\mathbb{R}$ is rigid: $\mathbb{R}$ has only one automorphism, its identity isomorphism. As a result, according to da Costa and Rodrigues [16], any notion of the scale of $\mathbb{R}$ is expressible, i.e., definable in the wide sense, in
the infinitary language associated with $\mathcal{R}$. However, since $\mathcal{R}$ is not denumerable, there are real numbers that are not definable in the (finitary) language of ZF plus the primitive symbols corresponding to primitive notions of $\mathcal{R}$.

We also have:

**Theorem 7.** Any structure can be extended to a rigid structure by the addition of a finite number of new primitive relations.

**Proof.** The proof of this theorem of Sebastião e Silva is presented in da Costa and Rodrigues [16].

The Galois group of a Riemannian geometry $V$ is usually trivial. So, in this case, any notion (element of the scale of $V$) is only definable in the wide sense in $V$. Moreover, there are notions that are not definable in the strict sense in $V$, with the help of the underlying language of ZF, as is clear.

## 5 Indiscernibility and related concepts

Since the beginning of quantum theory, in the early part of the twentieth century, notions such as indiscernibility and identity have been repeatedly discussed in the context of physics. For instance, Schrödinger argued, several times, that the notion of identity is meaningless in connection with elementary particles (see French and Krause [20] for a thorough discussion of this issue).

We now consider the analysis of the notions of indiscernibility, identity, and individuality in the domain of geometry. We take this to be a first step toward a future analysis of these concepts in the context of quantum physics. A key motivation here is the work done by Caulton and Butterfield [11].

Recall that our discussion is carried out in set theory, in which languages are both set-theoretic constructs and geometric structures. Geometry has, of course, two main faces, since there is a parallelism between geometry as a mathematical discipline and geometry as a physical theory underlying other physical theories. Both can be formulated in set-theoretically.

Identity, informally, constitutes the connection between objects $a$ and $b$ which are the same. We write, in this case, $a = b$. In set theory, ‘identity’ has two meanings: it may denote a basic concept of set theory or it may refer to the diagonal of the Cartesian product of a non-empty set by itself, i.e., a set-theoretic relation, formed by a set of ordered pairs whose two coordinates are the same. It is clear that, in set theory, infinitely many relations of identity can be expressed. In what follows, the reader will easily recognize the meaning in which term ‘identity’ is being employed.

Our set-theoretic ZF system is a pure set theory, in the sense that the only objects taken into consideration are sets. Identity, in the two set-theoretic senses above, can be defined or can be taken as a primitive notion. (On whether identity in general — understood as a fundamental concept rather than as a notion in set theory — can in fact be defined is an issue explored in Bueno [8] and [9].)

Geometric objects belong to scales of structures. So identity commonly refers to objects in certain geometric structures that are associated with languages. To talk about identity, in this case, is normally to talk about relations in the scales of structures, associated with abstract languages.
Identity is independent of language: if \( a \) and \( b \) are, for instance, figures in the scale of structure \( S \), then that \( a \) is identical to \( b \) (or is different from \( b \)) constitutes a relation that is independent of the language associated with \( S \). But this is not true of the notion of indiscernibility, which generalizes the concept of identity. We distinguish three kinds of indiscernibility: the weak, the strong, and the intrinsic kinds. They extended concepts presented in Caulton and Butterfield [11], although we use a different terminology.

5.1 Weak indiscernibility

By \( e = \langle D, r_i \rangle \) we denote, in the next definitions, a first-order structure in which \( r_i \) is a finite sequence, and \( L_e \) is the first-order finitary language associated with \( e \), and \( x \) and \( y \) are two distinct variables of \( L_e \). We then have the following definitions, in which \( r \) stands for any element of the sequence \( r_i \).

DEFINITION 8.
1. If \( r \) is a unary predicate the formula:
   \[ r(x) \leftrightarrow r(y) \]
   is called the companion formula of \( r \).
2. If \( r \) is a binary relation, then the formula:
   \[
   \forall z[(r(x, z) \leftrightarrow r(y, z)) \land (r(z, x) \leftrightarrow r(z, y))]
   \]
   is the companion formula of \( r \).
3. If \( r \) is a ternary relation, then
   \[
   \forall z \forall t[(r(x, t, z) \leftrightarrow r(y, t, z)) \land (r(t, x, z) \leftrightarrow r(t, y, z)) \land (r(t, z, x) \leftrightarrow r(t, z, y))]
   \]
   is the companion formula of \( r \).
4. The same goes for any \( n \)-ary relation \( r \), with \( n \) finite and greater than three.

Note that new variables \( t, z, \ldots \) are different from \( x \) and \( y \), and any two of them are distinct.

DEFINITION 9. \( x \equiv y \) abbreviates the conjunction of all companions of the elements of \( r_i \).

DEFINITION 10. If \( a \) and \( b \) are numbers of the domain of \( e = \langle D, r_i \rangle \), we say that they are weakly indiscernible (\( wi \)) if \( a \equiv b \) is true in \( e \).

Definitions 1, 2 and 3 can be modified to accommodate first-order languages and structures with infinitely many primitive relations, as well as infinitary languages. We now consider higher-order languages and structures.\(^4\)

DEFINITION 11. If \( e = \langle D, r_i \rangle \) is a higher-order structure with one of its associated languages, and \( a \) and \( b \) are elements of the scale of \( e \), then:

\(^4\)On higher-order model theory and infinitary model theory, which have connections with the general theory of structures, see, for example, Manin [30] and the references therein. Shelah insists that, in higher-order model theory, the central feature is not the language but certain algebraic properties of families of structures. In particular, it is not possible to study properly such structures in first-order languages.
1. If $a$ and $b$ have the same type and if there are companions of all or some of the relations $r_\iota$, then $a \equiv b$ is similarly defined as in the case of first-order languages and structures. Under these assumptions, $a$ is weakly indiscernible from $b$ if, and only if, $a \equiv b$ is true in $\varepsilon(D)$; otherwise, $x \equiv y$ is $x = x \land y = y$ and $a$ and $b$ are also said to be weakly indiscernible.

2. If $a$ and $b$ aren’t weakly indiscernible, then $a$ and $b$ are weakly discernible.

Here are some examples: (a) $\mathcal{R}^3$, with the standard metric $d$, constitutes a rigid structure whose Galois group is composed by the identity transformation. The same goes for the Klein geometry $K = \langle \langle \mathcal{R}^3, d \rangle, g \rangle$, in which $g$ is its group. As a result, in the associated infinitary languages, any two distinct points of $\mathcal{R}$ or of $K$ are weakly discernible. The same point applies to two distinct spheres.

(b) Any two points of an abstract projective plane are indiscernible (see, for instance, Heyting [22]).

5.2 Strong (or syntactic) indiscernibility

Let $a$ and $b$ be two figures of the same type (in particular, points) of the geometry à la Klein $e = \langle \langle D, r_\iota \rangle, g \rangle$. In this case, $a$ and $b$ are strongly discernible if there is a formula $F(x)$, of the language associated with $e$, which has $x$ as its sole free variable, such that, in $\varepsilon(D)$, it is true that:

$$\models F(a) \text{ and } \models \neg F(b).$$

If there is no such formula, $a$ and $b$ are said to be strongly indiscernible. It is clear that this definition can be extended to any structure, and depends on the language associated with it.

As for examples, we have: (1) In the usual higher-order, metric geometry of $\mathcal{R}^3$, two spheres, whose centers are strongly indiscernible, are also strongly indiscernible. (The language of the usual geometry is finitary and, as a consequence, there are real numbers that are strongly indiscernible.) (2) In the usual axiomatizations of Euclidean geometry, for example the one carried out by Hilbert, any two points are always strongly indiscernible.

5.3 Essential (or semantic) indiscernibility

In the Klein geometry $G = \langle \langle D, r_\iota \rangle, g \rangle$, any two figures $a$ and $b$ are said to be essentially indiscernible if there is a transformation of $g$ that maps $a$ onto $b$. Obviously, this definition can be extended to any structures whatsoever. When two figures of the scale of a structure are not essentially indiscernible, they are essentially discernible.

Here are a couple of examples: (1) In the metric geometry of $\mathcal{R}^3$, figures of the same type with domains essentially indiscernible are essentially indiscernible. (2) In the usual propositional calculus, considered as a free Boolean algebra, any two generators are essentially indiscernible.

Two other kinds of indiscernibility (or of discernibility), investigated by Caulton and Butterfield [11], can be extended to higher-order languages or to infinitary ones. However, we will not examine them here. We only note that Caulton and Butterfield’s absolute discernibility corresponds, in first-order languages, to our syntactic discernibility.
There are connections among our three concepts of discernibility (and indiscernibility) that hold for all mathematical structures. Using ideas of da Costa and Rodrigues [16], the following theorems can be proved:

THEOREM 12. In any structure \( e = (D, r_i) \), with the language \( L_e \), if two objects of \( e \) are syntactically discernible, then they are semantically discernible.

THEOREM 13. The converse of theorem 5.1 is not true in general.

THEOREM 14. Under the conditions of Theorem 5.1, if \( L_e \) is the strict infinitary language associated with \( e \) (see Section 1 above), then syntactic discernibility is equivalent to semantic discernibility.

Let \( e = (D, r_i) \) be a structure and \( L_e \) its language. An individual (or an individual of type \( i \)) is an element of \( D \) that is syntactically indiscernible from all other elements of \( D \). More generally, an individual of type \( t \) is an element of type \( t \) that is syntactically indiscernible from all other elements of type \( t \).

In some cases, the objects of type \( t, t \neq i \), can be the basic elements of the domain of a new structure and, in this new structure, they are individuals (of type \( i \) of the new structure). It is clear that the notion of indiscernibility depends not only on the structure we are considering, but also on the language employed.

In first-order languages, weak indiscernibility can be used as a kind of identity (see, for instance, Caulton and Butterfield [11]), although clearly it isn’t identity per se. The situation in higher-order structures and languages is entirely different. In this case, weak indiscernibility is too restrictive to have a similar role. In effect, the primitive objects of \( \varepsilon(D) \) that correspond to the structure \( e = (D, r_i) \) don’t have the same order and type, and \( r_i \) belongs to various types, so that weak indiscernibility becomes vacuous. Perhaps we could introduce weak indiscernibility for each type. However, it is better to use identity. But in the case of first-order structures or first-order languages, weak indiscernibility is relevant.

Most of the discussion in Caulton and Butterfield [11] that is devoted to first-order structures and languages can be extended to the higher-order case. For instance, in elementary Euclidean geometry, that is, Hilbert geometry, two spheres with the same radius are syntactically indiscernible. However, in the metric geometry of \( \mathbb{R}^3 \), any two spheres with radii of identical length are syntactically discernible when their centers are also syntactically discernible.

With regard to Caulton and Butterfield’s metaphysical views, their discussion can also be extended to the higher-order situation. To fix our ideas, we mention that their considerations on discernibility of pairs of objects are immediately generalizable to higher-order objects. In fact, several of their metaphysical views are adaptable to higher-order situations. In other words, their first-order metaphysics is extendable to a higher-order one. As an illustration, there can be higher-order individuals, similarly to the first-order individuals defined by them.

6 Structural metaphysics

In this section we outline a metaphysics of structures. Our preliminary analysis is intuitive and informal, but we will indicate how the approach can be
developed rigorously and formally. Our work is not exegetical, and we are not attempting to systematize the views defended by contemporary structural metaphysicians (see, for instance, Ladyman [29], French and Ladyman [19], and French [18]). Our aim is to present a possible metaphysics of structures inspired by these views.

Consider the notion of mathematical structure:

\[ e = \langle D, r_\iota \rangle \]

as introduced above. A particular case of structure is, obviously, that in which the sequence of primitive relations is empty:

\[ e = \langle D, \emptyset \rangle. \]

When \( D = \emptyset \), we have the empty structure that was not discussed above. When \( r_\iota \) is \( \emptyset \), the structure \( e \) can be considered as the set \( D \). So, from an informal stance, it seems reasonable to regard sets as a kind of structure.

Moreover, any relation is a strict relation, that is, an \( n \)-ary relation with \( n \geq 2 \), or a unary relation or set. Furthermore, relations are sets or, what is the same thing, unary relations. In particular, a strict \( n \)-ary relation, \( n \geq 2 \), is also a unary relation, that is, a set.

The structure:

\[ e = \langle D, r_\iota \rangle \]

is, basically, the binary relation:

\[ \{ \langle D, r_\iota \rangle \}. \]

Conversely, an \( n \)-adic relation \( r \), \( n \geq 2 \), essentially constitutes the structure:

\[ r^* = \langle K, r \rangle, \]

where \( K \) is a set \( \bigcup \{ \{ x \} : x \in r \} \).

As a result of these informal considerations, sets, relations and set-theoretic structures are, as it were, different faces of the same kind of basic objects: metaphysical structures or, to simplify, \( M \)-structures. Certain \( M \)-structures are fundamental: they are the \( M \)-sets, the bricks that compose all \( M \)-structures.

We can postulate, then, that \( M \)-sets constitute a system satisfying the axioms of a traditional set theory, such as ZFC, Zermelo-Fraenkel set theory with the axiom of choice (see Fraenkel and Bar Hillel [17], and Kuratowski and Mostowski [28]). However, the postulates of the theory should not include the axiom of regularity, given our assumption that any \( M \)-structure can be formed by other structures. In this way, our system reduces to ZF.

In addition to these considerations, it is natural to take ZF as providing a systematization of a metaphysics of \( M \)-structures, independently of its standard interpretation as a theory of usual sets or collections. However, we need to accommodate \( M \)-structures that are invoked in the representation of the physical universe (or, in a metaphysical reading, in the constitution of that universe). For this reason, we need to accommodate space and time. Accordingly, we add to ZF a new unary predicate \( \circ \), according to which \( \circ(x) \) means
that $x$ is a physical (or concrete) $M$-structure or $M$-Urelement (for example, particles may be conceived of as unary $M$-relations, or $M$-structures, etc.) The resulting system will be denoted by $ZF^c$. Postulates governing $M$-structures that satisfy $\sigma$ are added to those of $ZF$, depending on the view one has on the nature of the universe. We are also free to strengthen $ZF^c$ in various ways, for instance in order to handle $M$-categories and $M$-toposes interpreted as $M$-structures. (On $ZF$ with Urelemente, see Brignole and da Costa [5].)

We can now formulate standard scientific theories in an ontology of $M$-structures in analogy with the use of common set-theoretic structures in the foundations of science. In particular, fields and particles can be interpreted in terms of different kinds of $M$-structures. The philosophical simplicity that an ontology of $M$-structures brings to the philosophy of science, especially of physics, should be apparent.

Note that although $ZF^c$ was formulated in analogy with set-theoretic ideas, it does possess other interpretations, even non set-theoretic ones. This is the case of the pure $M$-structures, which can be used to capture some features of structuralist approaches in the philosophy of mathematics and science. $M$-structures can also be used to formulate Hughes’ suggestive idea of structural explanation. As he notes:

The idea of a structural explanation can usefully be approached via an example of special relativity. Suppose we were asked to explain why one particular velocity (in fact the speed of light) is invariant across the set of inertial frames. The answer offered in the last decade of the nineteenth century was to say that measuring rods shrank at high speeds in such way that a measurement of this velocity in a moving frame always gave the same value as one in a stationary frame. This causal explanation is now seen as seriously misleading: a much better answer would involve sketching the models of space-time which special relativity provides and showing that in these models, for a certain family of pairs of events, not only is their spatial separation $x$ proportional to their temporal separation $t$, but that the quantity $x/t$ is invariant across admissible (that is, inertial) coordinate systems; further, for all such pairs, $x/t$ always has the same value. This answer makes no appeal to causality; rather, it points out structural features of the models that special relativity provides. It is, in fact, a structural explanation. (Hughes [23], pp. 256-257.)

$M$-structures, suitably developed, provide a straightforward way of formulating the relevant models of special relativity and the invariances across various coordinate systems.

One need not, however, reify these invariant features of the models: they indicate traits that, according to the models under consideration, have a significant form of objectivity. It is not up to us that these invariant features emerge: they are stable properties of the relevant models. Of course, realists will read these traits as salient features of the world, whereas anti-realists will resist such interpretation. What matters to us is the understanding that emerges
from each interpretation and the fact that both can be captured in terms of $M$-structures. Explanation, including structural explanation, is clearly tied to understanding, and $M$-structures provide a rich framework to explore all of these alternatives. (We return to this point below.)

Realist views of structures insist that structures either provide us with ways of knowing the world (epistemic conceptions) or articulate ways the world is (ontic views). (For the distinction between epistemic and ontic views of structural realism, see Ladyman [29]; see also French [18] for additional discussion. For a critique of ontological conceptions of structure, see Arenhart and Bueno [1].) In contrast, anti-realist views of structure resist such commitments, and insist that structures, despite their help in systematizing information about the world, should not be reified. Due to its partiality, incompleteness, imprecision or inaccuracy, the information encoded in such structures is at best partial (for a thorough defense of the significance of partiality in this context, see da Costa and French [15]; see also Bueno [6]).

The proposal here is best thought of as being neutral between either side of the debate: it need not be committed to either realism about structures or to anti-realism about them. Structures can be used in geometry, physics or metaphysics quite independently of their particular interpretation, quite independently of the particular framework in which they are formulated. Set theory can be employed to formulate a $K$-geometry without stating that this geometry metaphysically is a particular set-theoretic structure. One can simply use set-theoretic resources as *representational tools* rather than as *constitutive devices*. That is, set theory provides a framework to represent and formulate various structures, but the framework itself need not be taken to answer the metaphysical question of the nature of the objects and structures that are represented in it. Nor is the framework indispensable. One can, after all, use different frameworks to carry out the task at hand: for instance, in addition to the many non-equivalent formulations of set theory, category theory, type theory, second-order mereology, among other frameworks, can all be employed as well. And in each of these frameworks, importantly different answers are given to the metaphysical question of the nature of the relevant structures: categorical, type-theoretic, mereological, etc. Since the metaphysical nature of the objects and structures is different in different frameworks, and all of these frameworks are possible, it’s unclear that any particular metaphysical answer is ultimately settled.

Even $M$-structures, despite being decidedly formulated in set theory, need not be taken to be *metaphysically* set-theoretic objects, since there are corresponding counterparts to such structures in all of the different frameworks mentioned above. On this interpretation, set theory is used as a representational tool only without any indication of the fundamental nature of the structures that are thus represented.

This distinction between set theory as a tool of representation and as a constitutive device can be used in other contexts in which set theory is employed. As is well known, Tarski often highlighted that he was a nominalist (for references and discussion, see Frost-Arnold [21]). Given all the work he did in set theory as well as his use of it in semantics and theories of truth, if set
theory were ontologically committed to sets as abstract objects (as the usual platonist reading would have it), Tarski would end up being incoherent. However, if set theory is simply interpreted as a representational tool, which need not capture or state the fundamental nature of the objects under study, one can use it while still resisting commitment to the existence of the objects and structures it refers to. (In addition, neutral quantifiers, which do not presuppose the existence of the objects that are quantified over, can also be invoked. Thus quantifying over sets is not enough to be ontologically committed to their existence: additional conditions of an existence predicate also need to be met; for details, see Azzouni [2] and Bueno [7].) This has the advantage of making Tarski’s stance coherent, while providing an alternative setting to understand the use of structures in a variety of domains, from geometry through physics to metaphysics.

We noted above the significance of structural explanations (and theoretical explanations) in physics. Are there such explanations in metaphysics? Clearly there should be. Certainly there are no causal explanations in this area, but still explanatory devices are systematically invoked. A metaphysical problem can be thought of as something that typically emerges from incompatible situations in (our conceptual description of) the world. Metaphysical explanations are then attempts to resolve the (apparent) tension. These explanations are considerations that indicate the possibility of certain situations in light of circumstances that seem to undermine them. For instance, how are individuals possible at the fundamental level given some interpretations of non-relativist quantum mechanics that deny that identity can be applied to quantum particles? How can abstract (that is, causally inert, non-spatial-temporal) mathematical structures exist given a world constituted by concrete (causally active, spatial-temporal) entities? Metaphysical explanations indicate that the apparent tension between these circumstances can be resolved. In some cases they suggest ways of reconciling the conflicting situations; in others, they provide reasons to undermine some of the built-in assumptions invoked in the description of the circumstances under consideration; in yet others, they suggest ways in which the inconsistency identified among the relevant situations can be tolerated.

In each case, some understanding emerges: we may realize the consistency of situations we thought were actually impossible; we may identify assumptions for which we may lack good reason to accept; we may learn about a broader range of possibilities (which may eventually include what was presumed to be, up to that point, impossibilities). In each case, our understanding increases, by gaining information about ways of reconciling what was thought to be incompatible, by obtaining information regarding false or questionable assumptions, or by increasing the scope of what is deemed possible. Metaphysical explanations are, thus, crucial for our understanding of several aspects of our conceptual framework. (For the general structure of philosophical explanations as ways of dispelling apparent tensions between conflicting assumptions, see Nozick [31].)

We think that $M$-structures provide a rich framework in which we can examine and assess these metaphysical explanations as they emerge in the foundations of the sciences. We can clearly identify the relevant assumptions: set-
theoretic, category-theoretic, type-theoretic, mereological, etc., depending on the context, as well as empirical, theoretical and conceptual presuppositions. We also probe each framework’s expressive and inferential power and its limitations. As a result, a proper assessment of their pros and cons can be determined and, in this way, some progress is eventually made. And our understanding, in the end, increases.

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