Second-order Logic Revisited

OTÁVIO BUENO*

In this paper, I shall provide a defence of second-order logic in the context of its use in the philosophy of mathematics. This shall be done by considering three problems that have been recently posed against this logic: (1) According to Resnik [1988], by adopting second-order quantifiers, we become ontologically committed to classes. (2) As opposed to what is claimed by defenders of second-order logic (such as Shapiro [1985]), the existence of non-standard models of first-order theories does not establish the inadequacy of first-order axiomatisations (Melia [1995]). (3) In contrast with Shapiro's suggestion (in his [1985]), second-order logic does not help us to establish referential access to mathematical objects (Azzouni [1994]). As I shall argue, each of these problems can be neatly solved by the second-order theorist. As a result, a case for second-order logic can be made. The first two problems will be considered rather briefly in the next section. The rest of the paper is dedicate to a discussion of the third.

1. Second-order Logic: Ontological Commitment and Non-standard Models

Quine has famously criticised second-order logic as nothing more than 'set theory in sheep's clothing' (Quine [1970], p. 66). In this sense, once we allow the quantification over predicate variables, we are committed to the whole set-theoretic hierarchy.

However, Boolos replied, second-order logic is clearly *weaker* than set theory (Boolos [1975], pp. 518-519). For although the notion of second-order validity can be defined in set theory, one cannot define, in purely second-order terms, the notion of set-theoretical truth. Thus, just by using second-order logic, one cannot be charged of being committed to set theory.

Moreover, Boolos argues, if we restrict ourselves to *monadic* second-order logic (which allows quantification only over monadic predicate variables), its ontological commitments do not go beyond those of first-order logic, given the introduction of plural quantifiers (see Boolos [1984] and [1985]). The idea is that the quantification over monadic predicates (such as 'is a critic') can be seen as a counterpart of a plural quantification in natural

* This paper has benefited from discussions with several people: Philip Catton, Steven French, Sarah Kattau, Bernard Linsky, Judy Pelham, Jaroslav Peregrin, Peter Simons, Göran Sundholm and Peter Woodruff. My thanks go to all of them.

language. So, instead of understanding the monadic second-order existential quantifier, $\exists X$, as 'there is a class', in Boolos's view, it can be read as the natural language plural quantifier 'there are (objects)'. For instance, the well-known Geach-Kaplan sentence, 'Some critics admire only one another', despite not being equivalent to any first-order sentence,[1] can be straightforwardly symbolised in second-order logic.[2] However, Boolos stresses ([1984], p. 449), in doing so we do not commit ourselves to the existence of additional items beyond those to which we are already committed. In order to understand this sentence (and, more generally, in giving a semantics for monadic second-order logic), we do not have to postulate, in addition to critics, a class of critics (or of whatever other objects we might be concerned with), and so our ontological commitments do not go beyond those of first-order logic.[3]

As a further argument in support of second-order logic, it has been pointed out that it is only in the context of second-order languages that mathematical practice can be properly represented (Shapiro [1985], [1990], and [1991]).[4] This is so for two main reasons: (i) Because of the expressive power of this logic, crucial notions—such as infinity, minimal closure and well-foundedness—which are not first-order definable, can be characterised in second-order logic. (ii) Because of the metatheoretical properties of second-order logic (especially the existence of categorical sets of second-order sentences with infinite models), it overcomes several shortcomings of weaker logics. In particular, it allows categorical representations of important mathematical theories (such as arithmetic and real analysis); that is, any two models of those theories, if formulated in a second-order language, are isomorphic. So, whereas first-order theories admit non-intended interpretations (precisely of arithmetic and real analysis), this is not the case for second-order ones. As a result, Shapiro argues, first-order axiomatisations are inadequate to accommodate mathematical practice.[5]

Despite these arguments, three main problems have been posed against second-order logic. Firstly, Resnik [1988] has criticised Boolos's approach on the grounds that, in natural language, plural quantifiers are understood in terms of *classes*: whatever understanding we have of these quantifiers, it is articulated out of our understanding of set-theoretic notions. Thus, according

---

[1] Kaplan's proof of this fact is presented in Boolos [1984], pp. 432-433.

[2] Supposing that the domain of discourse consists of critics, and Axy means 'x admires y', the Geach-Kaplan sentence becomes: $\exists X\,(\exists x\, Xx \wedge \forall x\, \forall y\, ((Xx \wedge Axy) \rightarrow x \neq y \wedge Xy))$.

[3] Boolos's idea is that the informal metalanguage, in which we give the semantics for (monadic) second-order logic, contains the plural quantifier 'there are objects', which is then used to interpret the second-order monadic quantifiers. (For further discussion, see Shapiro [1991], pp. 222-226.)

[4] A very helpful discussion of second-order logic can also be found in Peregrin [1997].

[5] An additional argument for second-order logic derives from the use of this logic by nominalists to reduce ontological commitment to mathematical objects. This is the case of Field's fictionalism and Hellman's modal-structural interpretation (see Field [1980] and [1989], and Hellman [1996]).

to Resnik, we do not have a grasp of plural quantifiers independently of the notion of class, and cannot assume them in the metalanguage of our semantics. So, as opposed to Boolos's view, by using these quantifiers, one does not avoid the ontological commitment to entities beyond first-order logic.

This criticism, however, is not decisive, since Boolos may insist that we *do* have a grasp of plural quantifiers independently of sets, namely via plural forms in natural language. And even if, in natural language, there are several distinct plural forms, Boolos has only assumed plural *quantifiers*, which are reasonably well understood. In discussing this point, Shapiro has suggested that, at this stage, the debate between Resnik and Boolos enters a regress *ad infinitum*, since 'Resnik might retort that even this [our understanding of plural quantifiers] is mediated by set theory, *first-order* set theory, in which case we have indeed entered the regress [...]' (Shapiro [1991], p. 226). But I think Boolos has still a point to make. Resnik's response disregards the fact that natural language (and, in particular, the plural forms it contains) has been articulated *previously* to set theory, and thus we cannot assume that our understanding of plurals (and, in particular, of plural quantification) depends upon set theory. Otherwise, Resnik would be committed to the counterintuitive claim that, before the development of set theory, we were unable to understand a sentence like 'Some critics admire only one another'. I do not deny, of course, that set theory may illuminate certain fragments of our language. I am only claiming that Boolos has a point in bringing plurals to bear on second-order quantification, since the former, at least in natural language, are well understood and have been previously introduced.

The second problem to the use of second-order logic, posed by Melia [1995], moves the discussion to the issue of categoricity. According to Melia, Shapiro [1985] has not shown that the existence of unintended interpretations of first-order theories establishes the inadequacy of first-order languages for formalising mathematics:

> If the mere existence of many non-isomorphic interpretations of first-order mathematical theories shows that first-order languages are unable to characterize certain mathematical systems, then the mere existence of many interpretations of second-order theories seems equally well to show that second-order languages are unable to characterize the relevant mathematical systems. (Melia [1995], p. 129)

But, again, a reply can be presented. Given that all interpretations of a certain second-order theory (say, arithmetic) are *isomorphic*—as opposed to what happens in the first-order case—the relevant mathematical system *is* properly characterised: after all, there is only one type of structure which satisfies it. In particular, it *does not matter* that, given the many interpretations, different objects may be assigned to the same 'position' in the structure.

For example, in the case of arithmetic, any object can be assigned to zero, provided it plays the right function in the natural number structure. What is relevant is the *kind* of structure defined—and the categoricity result guarantees that only *one* type of structure is forthcoming. This is, of course, the sort of structuralist view that Shapiro countenances (see Shapiro [1997]). Therefore, *contra* Melia, there is a huge difference between the many *non*-isomorphic interpretations of first-order mathematical theories and the many *isomorphic* interpretations of second-order ones.

The only argument Melia puts forward against this structuralist move is the following:

> The structuralist cannot say that the powerset of [a given set] *d* is simply that set which contains all the subsets of *d* for, on his view, *any* object can be a subset of *d*, providing it plays the right kind of role in a system which instantiates the right structure. The question: 'how are two mathematicians to know that they have the same structure in mind?' is unanswered. Without an account of how mathematicians can communicate the structure of the powerset, structuralists have no reason to think that the second-order quantifier provides a solution to any part of the communication problem. (Melia [1995], p. 132)

The problem with this argument is that it seems to confuse the structure of a mathematical *system* (namely, set theory) with the structure of a mathematical *object* (namely, the powerset). From the fact that any object (which satisfies the appropriate conditions) can be a subset of *d*, it does not follow that two mathematicians cannot know what structure they are talking about when they discuss set theory. The question whether they have the same (set-theoretical) structure in mind is settled by establishing an appropriate isomorphism between the structures they are considering. And the existence of this isomorphism is compatible with there being different 'interpretations' of the powerset. To communicate 'the structure of the powerset' (say, by associating a given object to it) is entirely *irrelevant* to communicate the structure of set theory itself. This is because, according to structuralism, mathematical objects (such as the powerset) are at best 'positions' in a structure—they lack any 'internal' structure. One of the points of structuralism is to argue that set theory (or any other mathematical theory for that matter) can be understood without any concern with the kind of objects which satisfy it.

The third problem for second-order logic was presented by Azzouni [1994]. It is also a reaction against Shapiro's view, and similarly to Melia's problem, it is concerned with the import of categoricity results for second-order logic. This problem is far more delicate than the ones considered thus far, and it shall be discussed in the next section.

2. Second-order Logic: How Not to Achieve Referential Access to Mathematical Objects

Azzouni is concerned with the problem of how we refer to mathematical objects, given that they are not (taken to be) subject to any causal constraints. He calls this problem the puzzle of referential access to mathematical objects (Azzouni [1994], p. 7). He then considers the claim that this problem can be solved by using second-order logic, since categorical definitions of certain mathematical structures become available in this context (Azzouni [1994], pp. 11-18). In resisting this claim, he has a clear target: Shapiro's view.

In his [1985] paper, Shapiro argues against the adequacy of certain logics to reflect mathematical practice. We have seen some of his arguments against first-order logic. In order to reject some further alternatives, Shapiro argues that a logic cannot presuppose the very mathematical objects whose grasping and understanding we are to accommodate. Thus, for instance, ω-logic (which is obtained from first-order logic by adding a new style of quantifier, ranging over natural numbers) cannot be taken as providing a solution to the issue of how we refer to natural numbers, since these numbers are presupposed in the formulation of this logic.[6]

According to Azzouni, the same sort of inadequacy is true of second-order logic. Indeed, in his view, 'ascending to second-order logic cannot solve the puzzle of referential access, if it is not already solved by merely invoking the standard interpretation in the first-order case' (Azzouni [1994], p. 12). In order to argue for this point, he considers a non-standard model theory for first-order logic (a truncated model theory) which has the same models as second-order logic with standard semantics. And, he argues, since no one would accept that, by means of this truncated logic (as Azzouni calls it), we would have solved the problem of referential access, no one is entitled to conclude that second-order logic has solved it as well. But what is this truncated logic?

The idea is to restrict the class of models of the first-order predicate calculus. If S is a set of statements in the language of this calculus, and A is the class of models of S, a *truncation* of A is a class of models of S which does not include every model in A. A *truncated model theory* is then a class of models which does not include every model. Such a theory is constructed by requiring, for instance, that a predicate symbol in the language hold only of certain sets (say, two-membered sets) in any model (Azzouni [1994], pp. 12-13). The crucial feature, Azzouni stresses, is that first-order logic with a truncated model theory has greater expressive power than the first-order

[6] As Shapiro remarks: 'the major shortcoming of ω-languages is that they assume or presuppose the natural numbers. Therefore, such a language cannot be used to show, illustrate, or characterize how the natural number structure is itself understood, grasped, or communicated' (Shapiro [1985], p. 733).

predicate calculus. However, we cannot claim to be able to solve the problem of referential access by adopting a truncation, after all:

> If I use a truncated model theory in which the natural numbers are 'truncated first-order definable', this certainly gives us no reason to believe that we have access to models of arithmetic that are isomorphic to the standard model—*for we have been given no explanation for how we are able to exclude the models not in the truncation.* (Azzouni [1994], p. 13; the italics are mine.)

Azzouni has to establish the equivalence between first-order logic with a truncated model theory and monadic second-order logic (Azzouni [1994], pp. 14-15). This is achieved by providing an appropriate method of translation between these logics. He first formulates the truncated model theory by admitting into the truncation only models M having the following properties: the domain of M is exhaustively and disjointly divided into two classes, A and B, such that A is an arbitrary set and B is the powerset of A. Moreover, two logical constants, 'S' and 'E', are introduced, and their references are fixed so that, in every model, 'S' is assigned to B, and 'E' is assigned to the set $\{(a, b): a \in \mathrm{A}, b \in \mathrm{B}, a \in b\}$. Finally, in every model, individual constants are assigned to A, and the extension of n-place predicates is a subset of A^n.

Azzouni then shows how to translate sentences of monadic second-order logic into the language of the truncated logic. For instance, the second-order induction scheme for Peano's arithmetic

$$\forall X\, ((X0 \wedge \forall x\, (Xx \rightarrow Xsx)) \rightarrow \forall x\, Xx)$$

becomes

$$\forall x\, (\mathrm{S}x \rightarrow ((\mathrm{E}0x \wedge \forall y\, (\mathrm{E}yx \rightarrow \mathrm{E}syx)) \rightarrow \forall y\, \mathrm{E}yx)).$$

The idea is, of course, to consider this version of second-order logic as a two-sorted first-order logic, eliminating the additional quantifiers by relativising them to predicates.

The last step is to provide a one-one mapping of monadic second-order logic onto the truncated model theory. Azzouni's definition is as follows (Azzouni [1994], p. 15). A model A_s of second-order model theory is mapped to a model A_t of truncated model theory in such a way that:

(i) If d_s is the domain of A_s, then d_t is the union of d_s and of the powerset of d_s.
(ii) 'S' is mapped to the powerset of d_s.
(iii) 'E' is mapped to the set $\{(a, b): a \in d_s, b \in \text{powerset of } d_s, a \in b\}$.

(iv) Individual constants are mapped to the same items in d_t as they are mapped to in d_s. Moreover, *n*-place predicates are mapped to the same subset of d_t as they are in d_s.

As a result, Azzouni notices (*ibid.*, p. 15), the truncated logic has exactly the same metalogical properties as monadic second-order logic (with standard semantics). In particular, completeness, compactness and Löwenheim-Skolem do not hold; both logics have the same capacity to characterise infinite structures, and they have the same models. In this sense, they are equivalent.

The moral Azzouni draws from this equivalence is that there is no warrant to claim that second-order logic solves the accessibility problem, since it is 'a simple act of fiat to fix the language and the model theory in the way I have to get [...] truncated logic' (Azzouni [1994], p. 16). But then someone may wonder why, in the case of second-order logic with standard semantics, we do *not* end up with any impression of fiat. According to Azzouni, this is because 'the acts of fiat are neatly tucked away in the second-order quantifier and the grammatical relation of predicate constant and variable to individual constant and variable' (*ibid.*, p. 17). Here is his argument:

> Normally, first-order logic is not taken to commit one to the extensions of the predicates as entities over and above what such extensions are composed of, but only to what the quantifiers range over. Thus the notation '*Pa*', which is a one-place predicate symbol syntactically concatenated with a constant symbol, is not taken to contain an (implicit) representation of $\in$.
>
> However, as soon as we allow ourselves to quantify (standardly) into the predicate position, this is precisely how syntactic concatenation *must be* understood. Furthermore, and this is striking (or ought to be), syntactic concatenation in these contexts is *not* open to reinterpretation across models—it is an (implicit) logical constant.
>
> This explains, I think, [...] why, when we make the concatenation involved here explicit in terms of 'E', it is clear that 'E' must be fixed in its interpretation across models (and one naturally observes that doing so presupposes, to some extent, our grasp of $\in$) [...]. (Azzouni [1994], p. 17)

Thus, in Azzouni's view, the second-order terminology and notation neatly masks the fact that, when using second-order logic, we are presupposing a grasp of certain mathematical notions (such as the membership relation of set theory). Hence, Shapiro has to reject this logic on nearly the same grounds as those he gives in rejecting ω-logic, and withdraw his claim that second-order logic offers a solution to the problem of referential access.

But how compelling are Azzouni's arguments?

3. Second-order Logic: Achieving Referential Access to Mathematical Objects

In my view, the arguments Azzouni put forward are far from being decisive. First, notice that the truncation was obtained, in a clearly *ad hoc* way, *assuming the information provided by second-order logic* about which models to exclude from the class of first-order models. It is because we have second-order logic at our disposal that we are able to construct the equivalent truncated logic (the former provides heuristic guidelines for the construction of the latter). So, Azzouni can only maintain the adequacy of his construction if he *assumes that second-order logic solves the problem of referential access* in the first place. But if he grants this point (as he should, in order to have his case), the second-order theorist can simply reject the truncation on the basis of its *adhocness*. Far from establishing the inadequacy of second-order logic to accommodate the referential access, the truncation spells out the advantage of this logic, since it allows us to clearly distinguish between an *ad hoc* construction (which assumes second-order logic, but only mimic its features) and an adequate, original one (the class of categorical theories achieved through the proper resources of the second-order language).

Secondly, Azzouni's explanation of why we do not have the impression of fiat when we use second-order logic misunderstands the nature of second-order quantifiers. His claim that 'as soon as we allow ourselves to quantify (standardly) into the predicate position', 'syntactic concatenation *must be* understood' as 'an (implicit) representation of $\in$', simply does not hold. As we saw, Boolos introduced plural quantification precisely to avoid commitment to classes in interpreting (monadic) second-order quantifiers. In fact, he has shown (in Shapiro's words), 'how a single predicate R (in the metalanguage) can code an assignment of "values" to the second-order variables of the formal language' (Shapiro [1991], pp. 223-224). The idea is that $\langle V, v\rangle$ is in R iff V is a second-order variable and v are the objects to be assigned to V. According to Boolos, a second-order formula of the form Vv is then interpreted thus: we say that R and an assignment s to the first-order variables (only) satisfy Vv iff $R\langle V, s(v)\rangle$ (where '$\langle,\rangle$' is the ordered-pair function sign; see Boolos [1985], p. 336). Thus, there is no need for understanding concatenation in terms of set-theoretical membership. Moreover, any impression of fiat can be explained not set-theoretically, but by an appropriate understanding of (monadic) second-order quantifiers, such as that provided by Boolos's approach.[7]

[7] Exploring the fact that full second-order logic is impredicative, professor Sundholm presented a beautiful argument that indicates the difficulty to explain the meaning of the second-order quantifier in an impredicative context (see Sundholm [1998]). The natural move for the second-order theorist to make is to go for *ramified* second-order logic. In this setting, relations are only defined in levels, and once a relation is defined at a particular level, it can then be used in defi-

Moreover, Azzouni's remark that 'syntactic concatenation in these [second-order] contexts is *not* open to reinterpretation across models—it is an (implicit) logical constant' can be explained straightforwardly, in Boolos's approach, with *no* commitment to $\in$. The idea is simply that, as we saw in the previous paragraph, syntactic concatenation is accommodated by the ordered-pair function. But since this function has fixed properties, which do not change across models, it does not allow syntactic concatenation to be open to reinterpretation.

Now, Azzouni's trick in the construction of the truncated model theory is to select those models such that 'E' is interpreted as '$\in$'. As a result, sentences of the form Xx become $x \in X$, which means that second-order predication is interpreted as membership. This move can be traced back to Quine's [1970] criticism of second-order logic. Quine's advice to the logician was to avoid second-order quantification and adopt a formulation in terms of membership; that is, instead of writing '$\exists X\, Xa$', he or she should write '$\exists \alpha\, a \in \alpha$'. Now, the problem with this suggestion, as Boolos [1975] has stressed, is that it is neither validity-preserving nor implication-preserving. For instance, although '$\exists X\, \forall x\, Xx$' is valid, the same is not the case for '$\exists \alpha\, \forall x\, x \in \alpha$'. Moreover, although '$x = z$' follows from '$\forall Y\, (Yx \leftrightarrow Yz)$' by logic alone, it does *not* follow from '$\forall \alpha\, (x \in \alpha \leftrightarrow z \in \alpha)$' only by logic, but it requires some set theory. It now becomes clear why second-order quantification and set-theoretical membership are indeed quite distinct. And with this distinction in hand, we can reject Azzouni's insistence that second-order logic assumes set theory.

Furthermore, notice that, in Azzouni's construction of the truncated model theory, the introduction of the two predicates, 'E' and 'S', has the role of 'extending' the first-order quantifiers in such a way that we achieve the expressive power of second-order logic *without* actually using second-order quantification. Now, in the standard semantics for second-order logic, there are no restrictions on the sets one considers when interpreting the range of second-order variables. And this is crucial if we are to make second-order logic *really* second-order. In order to make *first*-order logic resemble second-order, Azzouni has to introduce 'E' and 'S', which are to be read respectively as 'is a member of' and 'is a set'. In other words, in order to achieve the expressive power of second-order languages in a first-order setting, Azzouni *has to introduce set-theoretic notions*. But *without any recourse to this notions*, second-order logic is able to achieve its expressive power. So the most we can say Azzouni has established is that, without set-theoretic talk, *first*-order logic lacks the expressive power of second-order logic. Would this prove that

nitions at later levels. In this way, by avoiding impredicative definitions, the meaning of the ('ramified') second-order quantifier can be explained.

the latter already *presupposes* set theory? Surely not! As we saw, the *point* of Boolos's plural interpretation of second-order quantifiers is precisely to establish that we are not committed to set-theoretic talk in order to provide a semantics for second-order logic.

I conclude, thus, that for all he has said, Azzouni has not established the inadequacy of second-order logic to solve the problem of referential access, and hence Shapiro can still maintain his case. Moreover, given the previous rejection of the problems posed by Resnik and Melia, I take it that second-order logic, even if not unshakeable, is more robust than the above critics have supposed.

Otávio Bueno
Division of History and Philosophy of Science
School of Philosophy
University of Leeds
Leeds, LS2 9JT, UK
phloab@leeds.ac.uk

References

Azzouni, J. [1994]: *Metaphysical Myths, Mathematical Practice: The Logic and Epistemology of the Exact Sciences.* Cambridge: Cambridge University Press.

Boolos, G. [1975]: 'On Second-Order Logic', *Journal of Philosophy* 72, pp. 509-527.

Boolos, G. [1984]: 'To Be is to Be a Value of a Variable (Or to Be Some Values of Some Variables)', *Journal of Philosophy* 81, pp. 430-449.

Boolos, G. [1985]: 'Nominalist Platonism', *Philosophical Review* 94, pp. 327-344.

Field, H. [1980]: *Science without Numbers: A Defense of Nominalism.* Princeton, N.J.: Princeton University Press.

Field, H. [1989]: *Realism, Mathematics and Modality.* Oxford: Basil Blackwell.

Hellman, G. [1989]: *Mathematics without Numbers: Towards a Modal-Structural Interpretation.* Oxford: Clarendon Press.

Melia, J. [1995]: 'The Significance of Non-Standard Models', *Analysis* 55, pp. 127-134.

Peregrin, J. [1997]: 'What Does One Need, When She Needs "Higher-Order Logic"?', in *Proceedings of Logica'96*, Prague: Filosofia, pp. 75-92.

Quine, W.V.O. [1970]: *Philosophy of Logic*. Englewood Cliffs, N.J.: Prentice-Hall.

Resnik, M. [1988]: 'Second-order Logic Still Wild', *Journal of Philosophy* 85, 75-87.

Shapiro, S. [1985]: 'Second-order Languages and Mathematical Practice', *Journal of Symbolic Logic* 50, pp. 714-742.

Shapiro, S. [1990]: 'Second-order Logic, Foundations, and Rules', *Journal of Philosophy* 87, pp. 234-261.

Shapiro, S. [1991]: *Foundations Without Foundationalism: A Case for Second-order Logic*. Oxford: Clarendon Press.

Shapiro, S. [1997]: *Philosophy of Mathematics: Structure and Ontology*. New York: Oxford University Press.

Sundholm, G. [1998]: 'Intuitionism and Logical Tolerance', forthcoming in *Vienna Circle Institute Yearbook* 6.