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A Defense of Second-Order Logic

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Abstract Second-order logic has a number of attractive features, in particular the strong expressive resources it offers, and the possibility of articulating categorical mathematical theories (such as arithmetic and analysis). But it also has its costs. Five major charges have been launched against second-order logic: (1) It is not axiomatizable; as opposed to first-order logic, it is inherently incomplete. (2) It also has several semantics, and there is no criterion to choose between them (Putnam, J Symbol Logic 45:464–482, 1980). Therefore, it is not clear how this logic should be interpreted. (3) Second-order logic also has strong ontological commitments: (a) it is ontologically committed to classes (Resnik, J Phil 85:75–87, 1988), and (b) according to Quine (Philosophy of logic, Prentice-Hall: Englewood Cliffs, 1970), it is nothing more than "set theory in sheep's clothing". (4) It is also not better than its first-order counterpart, in the following sense: if first-order logic does not characterize adequately mathematical systems, given the existence of *non-isomorphic* first-order interpretations, second-order logic does not characterize them either, given the existence of *different* interpretations of second-order theories (Melia, Analysis 55:127–134, 1995). (5) Finally, as opposed to what is claimed by defenders of second-order logic [such as Shapiro (J Symbol Logic 50:714–742, 1985)], this logic does not solve the problem of referential access to mathematical objects (Azzouni, Metaphysical myths, mathematical practice: the logic and epistemology of the exact sciences, Cambridge University Press, Cambridge, 1994). In this paper, I argue that the second-order theorist can solve each of these difficulties. As a result, second-order logic provides the benefits of a rich framework without the associated costs.

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1 Introduction

There are logics towards which one cannot be indifferent. One can accept them, or reject them, but one cannot be indifferent between these two options. Second-order logic is one of these logics. It has received considerable attention in the last few years, due to its peculiar status. In contrast with *n*th-order logics ($n \geq 3$), second-order logic is still relatively close to first-order logic, and thus its ontological commitments are still relatively weak. However, as opposed to first-order logic, second-order logic allows quantification over relations and functions of the language. This provides a formidable expressive power, which goes far beyond what can be expressed in first-order languages. But it also opens up the possibility of criticism.

Five major criticisms have been launched against this logic: (1) It is not axiomatizable; as opposed to first-order logic, second-order logic is inherently incomplete. (2) It also has several semantics, and there is no criterion to choose between them Putnam (1980). Therefore, it is not clear how this logic should be interpreted. (3) Second-order logic also has strong ontological commitments: (a) it is ontologically committed to classes Resnik (1988), and (b) according to Quine (1970), it is nothing more than 'set theory in sheep's clothing'. (4) It is also not better than its first-order counterpart, in the following sense: if first-order logic does not characterize adequately mathematical systems, given the existence of *non-isomorphic* first-order interpretations, second-order logic does not characterize them either, given the existence of *different* interpretations of second-order theories Melia (1995). (5) Finally, as opposed to what is claimed by defenders of second-order logic [such as Shapiro (1985)], this logic does not solve the problem of referential access to mathematical objects Azzouni (1994).

These criticisms are, of course, quite varied in their focus: the first two raise *metalogical* worries (about incompleteness and the existence of various semantics for second-order logic); the third brings an *ontological* issue (concerning second-order logic's involvement with set theory); the fourth concerns a *methodological* point: the adequacy of the models of second-order theories; and the fifth addresses an *epistemological* difficulty with referential access to mathematical objects in second-order settings. If these criticisms were correct, serious problems would be faced by second-order logic. In this paper, I argue that the second-order theorist can solve each of these difficulties.

2 On Second-Order Logic

Before examining these problems, I will provide some background information on second-order logic that will be used in the discussion that follows. As indicated, the main feature of second-order languages is that they allow quantification over predicate and function variables. So, '$\exists X\ \forall x\ Xx$' and '$\exists f\ \exists x\ \exists y\ f(x) = y$' are sentences of these languages. Now, second-order logic can be presented by a clear formal system, which is a natural extension of the system for classical first-order logic. Besides the well-known axioms for first-order logic, the following axioms and rules are added (see Shapiro 1991, pp. 65–70):

$\forall X^n\ \varphi(X^n) \rightarrow \varphi(T)$, where T is either an n-place relation variable free for X^n in φ or a non-logical n-place relation letter.
$\forall f^n \varphi(f^n) \rightarrow \varphi(p)$, where p is either an n-place function variable free for f^n in φ or a non-logical n-place function letter.
From $\varphi \rightarrow \psi(X)$ infer $\varphi \rightarrow \forall X\ \psi(X)$, provided that X is not free in φ or in any premise of the deduction.
From $\varphi \rightarrow \psi(f)$ infer $\varphi \rightarrow \forall f\ \psi(f)$, provided that f is not free in φ or in any premise of the deduction.
(*Axiom schema of comprehension*) $\exists X^n\ \forall x_1 \ldots \forall x_n\ (X^n x_1 \ldots x_n \leftrightarrow \varphi x_1 \ldots x_n)$, provided that X^n is not free in φ.[1]

Second-order logic also has a straightforward semantics, which is an extension of the semantics for first-order logic. It is called *standard* semantics (see Shapiro 1991, pp. 70–76). Similarly to first-order semantics, a *model* of second-order logic is a structure $\langle D, I \rangle$, where D is a non-empty set and I is an *interpretation function*, which assigns appropriate items obtained from D to the non-logical terminology of the language in question. For example, if c is an individual constant, $I(c)$ is an element of D. If P is a three-place relation symbol, $I(P)$ is a subset of $D \times D \times D$. The notion of *variable assignment*—a particular function defined over the variables of the language—is also an extension of the corresponding first-order notion. It assigns (1) an element of D to each first-order variable of the language, (2) a subset of D^n to each n-place relation variable, and (3) a function from D^n to D to each n-place function variable. A crucial addition comes with the concepts of denotation and satisfaction which incorporate, besides the usual conditions for first-order components of the language, also conditions for second-order variables (Shapiro 1991, p. 72):

If $M = \langle D, I \rangle$ is a model and s an assignment on M, the *denotation* of $f^n t_1 \ldots t_n$ in M, s is the value of the function $s(f^n)$ at the sequence of members of D denoted by the members of $t_1 \ldots t_n$.
If X^n is a relation variable and $t_1 \ldots t_n$ is a sequence of n terms, then $M, s \mid= X^n t_1 \ldots t_n$ if the sequence of members of D denoted by the members of $t_1 \ldots t_n$ is an element of $s(X^n)$.
$M, s \mid= \forall X\ \varphi$ if $M, s' \mid= \varphi$ for every assignment s' that agrees with s at every variable except possibly X.
$M, s \mid= \forall f\ \varphi$ if $M, s' \mid= \varphi$ for every assignment s' that agrees with s at every variable except possibly f.

This semantics makes second-order logic properly second-order: the Löwenheim-Skolem theorems do not hold, and the logic is neither compact nor complete;

[1] The second-order existential quantifier is defined in the usual way: $\exists X\ \varphi =_{\text{def.}} \neg\forall X\ \neg\varphi$, and $\exists f\ \varphi =_{\text{def.}} \neg\forall f\ \neg\varphi$.

moreover, theories with infinite domains are adequately characterized (see Shapiro 1991, pp. 80–88).

It turns out that this is not the only semantics for second-order logic. The latter also has what is called *Henkin* semantics (see Shapiro 1991, pp. 73–74). The crucial feature of this semantics is that the relation and function variables do not range over *all* the relations and functions on the domain, but only over a fixed collection of them. A *Henkin model* is then a structure $M^{H} = \langle D, R, F, I\rangle$, where D and I are, respectively, the domain and the interpretation function, R is a sequence of collections of relations, and F is a sequence of collections of functions. Moreover, for each n, $D(n)$ is a non-empty subset of the powerset of D^n, and $F(n)$ is a non-empty collection of functions from D^n to D. The idea is that $F(n)$ is the range of n-place function variables and $D(n)$ of n-place relation variables. In this setting, a *variable assignment* is a function which assigns, to each first-order variable, a member of D; to each n-place relation variable, a member of $D(n)$; and to each n-place function variable, a member of $F(n)$. The remaining conditions are the same as those presented above, except that the notions of denotation and satisfaction are defined for a Henkin model $M^{H} = \langle D, R, F, I\rangle$, rather than a (standard) model $M = \langle D, I\rangle$.

The consequence is that, given the restrictions imposed by R and F, with Henkin semantics second-order logic becomes complete, compact and Löwenheim-Skolem theorems do hold (see Shapiro 1991, pp. 88–95). But these results only hold with an important qualification: appropriate Henkin models have to be considered in order for Henkin semantics to be *sound*. In fact, it is not difficult to establish that some Henkin models do *not* satisfy the comprehension schema of second-order logic. Let $M^{H} = \langle D, R, F, I\rangle$ be a structure such that D is a set with at least two members $a \neq b$; $R(2)$ has a single member, $\{\langle x, a\rangle : x \in D\}$; and $F(1)$ has also a single member, the function whose value is x for every $x \in D$. Therefore, M^{H} does *not* satisfy the sentence $\exists X\ \forall x\ \forall y\ (Xxy \leftrightarrow x \neq x)$, which is an instance of the comprehension schema. This sentence asserts the existence of an empty binary relation; but *no* such a relation is found in M^{H}. As a result, the notion of a Henkin model has to be restricted still further: in such a way that only those Henkin models that satisfy the comprehension principle, as well as the remaining second-order axioms, are considered. Such models are called *faithful* (see Shapiro 1991, pp. 88–89).

We can now examine each of the problems that have been raised against second-order logic, evaluating their weaknesses and strengths.

3 Metalogical Worries: Incompleteness and 'Non-Standard' Semantics

An important reason why second-order logic is viewed with suspicion derives from its non-axiomatizability. Since there is no complete proof procedure for this logic, this is used as an argument against it, assuming that something can be taken as a logic *only if it is complete*. But why is completeness taken to be a decisive feature of a logical system? Bluntly put, because it establishes that all valid sentences are derivable, and one does not want to face the predicament of not being able to derive certain valid sentences.

The problem with this argument is that it disregards an important point. If first-order logic is complete, this is in part due to its weak expressive power. Several sentences are not (and cannot be) valid in first-order logic simply because they are not expressible in it. This is the case, for example, of '$\exists X\, \forall x\, Xx$', which is valid in second-order logic, but not expressible in a first-order language. Moreover, as is well known, several mathematical notions are not first-order characterizable. The ordered field $\Re$ of real numbers is a case in point, since there is no first-order set of axioms which characterize $\Re$ up to isomorphism [see, for instance, Barwise (1977a, b, pp. 11–12)]. In other words, despite being part of the daily currency of mathematical practice, the field $\Re$ cannot be characterized in first-order languages. So, when an analyst is proving that certain axioms characterize $\Re$ up to isomorphism, he or she has already gone beyond first-order resources. Furthermore, as is also well known, not even the notion of identity can be expressed in first-order logic: it has to be taken as primitive. (Of course, in second-order logic, this notion can be straightforwardly expressed: $x = y \leftrightarrow \forall P\, (Px \leftrightarrow Py.)$[2] The point here is that the advantage of having completeness in first-order logic is not decisive since the expressive power of this language is too limited. Why should we take completeness as something so important if so many significant concepts cannot be expressed in a complete logic? What is the rationale behind this requirement?

A further point emerges when we consider the importance of categoricity for an adequate characterization of mathematical theories. In order to develop an interpretation of mathematics, it is crucial to have the resources to characterize mathematical structures properly. A necessary condition for this is to determine those structures up to isomorphism. In other words, there should be a function from the domain of a structure A onto the domain of a structure B such that this function is bijective and preserves the relations that hold for the objects in A and B. The nature of these objects is arguably immaterial for the understanding of mathematical discourse. What matters is the structure that is thus determined.

One of the crucial features of second-order languages is the fact that they allow for categorical characterizations of theories with infinite domains. In particular, each of arithmetic and real analysis is categorical in a second-order setting (that is, any two models of either theory, if formulated in a second-order language, are isomorphic). So, whereas first-order theories admit non-intended interpretations (e.g. in arithmetic and real analysis), this is not the case for second-order theories. As is well known, set theory (say, Zermelo set theory, Z) is not in general categorically characterized in this way. However, it is possible to show that if Z is formulated in second-order terms (Z^2), given two models of Z^2, if they are not isomorphic, one of them is isomorphic to what may be called an "initial segment" of the other (see Shapiro 1991, p. 86). In this sense, Z^2 is *quasi-categorical*. Moreover, as Montague (1965) has argued, in a second-order framework, the notion of *standard model* of set theory can be properly characterized, since in this setting the models of set theory are such that '$\in$' is mapped to membership, there are sets with the adequate cardinalities, and so on. Thus, as opposed to what happens in

[2] Shapiro provides further examples of mathematical concepts that can only be adequately expressed in second-order languages (1991, pp. 97–109).

first-order languages, there is a clear sense in which models of second-order theories *are* standard.

Finally, in second-order languages, an impressive amount of mathematics can be expressed. Virtually everything that is needed to formulate contemporary mathematics (excluding, of course, higher set theory) and all the mathematics required for applications to science can be expressed in second-order languages. The theory known as PA^2 (Peano Arithmetic in second-order logic) provides this conceptual framework (see Hellman 1989; Boolos 1998). In fact, as Shapiro stressed, it is only in the context of second-order logic that mathematical practice can be properly represented (Shapiro 1985, 1990, 1991). Because of the expressive power of this logic, crucial notions—such as infinity, minimal closure, and well-foundedness—which are not first-order definable, can be characterized in a second-order language. Moreover, because of the metatheoretical properties of second-order logic (especially its categoricity results), it overcomes several shortcomings of weaker logics. As a result, second-order axiomatizations are adequate to accommodate mathematical practice (as opposed to first-order ones).[3]

My point here is twofold: first, to stress the expressive power of second-order languages, and second, to highlight the fact that several important mathematical theories formulated in a second-order setting are categorical (or, at least, quasi-categorical). Thus, the relevant structures are characterized up to isomorphism, and in this sense they are formulated in a better way than in first-order languages. In other words, in a first-order setting, (1) either we cannot formulate certain mathematical notions and theories at all, or (2) if we can, all those theories with infinite domains cannot be characterized properly (given the existence of non-intended interpretations). As opposed to this, in a second-order setting, (1′) we are able to formulate notions beyond the limits of first-order languages, *and* (2′) many important theories with infinite domains can be characterized categorically.

Once these points are presented, it becomes clear that the issue of the completeness of a logical system should be examined in a broad, holistic context. Completeness is not a feature of a logic to be absolutely preferred to any other feature. In this respect, it differs from *soundness*, which can be taken as a minimum adequacy condition for a logic. For a system that allows us to derive invalid sentences—i.e. an *unsound* system—is *not* reliable, and therefore it makes good sense to have reservations about it.

Now, besides completeness, which other features can a logical system have? It can be (a) decidable, (b) categorical (it makes certain theories categorical), and (c) rich in expressive power. It is in the broad context of *these* three features that we should evaluate the desirability of completeness. First- and second-order logics fare in exactly the same way with regard to the first feature, and second-order logic fares

[3] As a further argument for second-order logic—at least for those with nominalist inclinations!—note that several nominalist proposals have adopted versions of this logic as part of their strategies to reduce ontological commitment to mathematical objects. This is the case of Field's fictionalism [at least as presented in Field (1980), see Field (1989) for a first-order version] and Hellman's modal-structural interpretation (Hellman 1989).

far better regarding the other two. Indeed, neither of these logics are decidable in general, but in both cases their monadic fragments are.[4] Thus, the situation between these logics is the same vis-à-vis this issue. However, once we consider categoricity and expressive power, the situation changes entirely. As we saw, as opposed to first-order theories, those formulated in second-order languages can be categorical, and also have stronger expressive power. It now becomes clear that, in the context of *these* features, the lack of completeness is no longer a decisive criticism of second-order logic, and is certainly outweighed by the categoricity, expressibility and manageability of this logic.[5]

We can now examine a different kind of argument against second-order logic that has been presented by Putnam (1980). Since Putnam's argument was formulated as part of a broader context, and was not explicitly meant to be a criticism of second-order logic, let me say a few words about this context. Putnam's criticism emerged as part of his defense of what is called the model-theoretic argument against metaphysical realism. The point of this argument is to show that metaphysical realism is incoherent. According to the metaphysical realist picture, truth and justification should be distinguished, since truth is radically non-epistemic: our best-supported theories can be false. In fact, even our *ideal* theory (a theory that is, e.g., consistent, complete, empirically adequate, simple, and plausible) *can still be false*. What Putnam's model-theoretic argument establishes is that, by using the Löwenheim-Skolem theorem, the ideal theory *comes out true*. Therefore, Putnam concludes, "the metaphysical realist's claim that even the ideal theory might be false 'in reality' seems to collapse into unintelligibility" (Putnam 1980, p. 13).[6]

I am not concerned with the details of the model-theoretic argument here. Note only that it is crucial for (this version of) Putnam's argument to use the existence of several non-isomorphic interpretations of a given (first-order) theory. The important point for our purposes is that Putnam discusses a critique of his argument that relies on second-order logic. In a second-order context, we have categoricity results and such non-isomorphic interpretations are not available. Hence Putnam's model-theoretic argument can be blocked using second-order logic.

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[4] Monadic first-order logic has predicates with only one place, while monadic second-order logic has one-place predicates and second-order quantifiers binding one-place predicate variables.

[5] Further arguments in support of this point can be found in Shapiro (1991) and Boolos (1975).

[6] Here is Putnam's argument (1976, pp. 125–126). Let us assume that T is an ideal theory. It is consistent, complete, empirically adequate, simple, plausible etc. The only property of T that is left open (for the sake of argument) is its truth. (T may well be false.) Now, let us assume, with Putnam, that THE WORLD can be broken into infinitely many pieces, and that T states that there are infinitely many things. Since T is consistent, by the completeness theorem of first-order logic, it has a model. And since T has an infinite model, by the Löwenheim-Skolem theorem, it has a model of every infinite cardinality. Let us now select a model M that has the same cardinality as THE WORLD, and let us establish a one-to-one mapping from individuals of M into pieces of THE WORLD. We can use this mapping to define relations over M's domain directly in THE WORLD, and as a result, we can define a satisfaction relation, SAT, which establishes a 'correspondence' between T's language and sets of pieces of THE WORLD. Now, provided that we interpret 'true' as TRUE(SAT)—that is, as the truth predicate that is defined by the relation SAT, just as 'true' is defined in terms of 'satisfies' in Tarski's account—the ideal theory T comes out *true*: indeed, it is true of THE WORLD. But then T cannot be false!

What matters here is Putnam's reply to this criticism. As he points out Putnam (1980, p. 23), second-order logic not only has a standard semantics—according to which second-order logic is incomplete, Löwenheim-Skolem theorems do not hold, and it has categorical models—but it also has, as we saw above, a Henkin semantics, and according to *this* semantics second-order logic *is* complete, Löwenheim-Skolem theorems *do* hold, and it lacks categorical models. The crucial feature of Putnam's reply is that we have no way of choosing between these two semantics. In his own words:

> the 'intended' interpretation of the second-order formalism is not fixed by the use of the formalism (the formalism itself admits so-called 'Henkin models', i.e., models in which the second order variables fail to range over the *full* power set of the universe of individuals). (Putnam 1980, p. 23)

Therefore, we cannot claim that the categoricity results hold in second-order logic: *that depends upon the semantics we adopt*.

Now, this argument of Putnam's could be presented in a broader context, independently of the discussion of his model-theoretic argument. And in this context, Putnam's reply could be used as an argument for the inadequacy of second-order logic, since we presumably have no criteria to choose between these two semantics, and therefore we do not know what are the metalogical properties of this logic.

But the second-order theorist has something to say here. First, note that if we take *all* Henkin models to provide a semantics for second-order logic, the resulting semantics is *not even sound*, since as we saw, the comprehension schema of this logic is *not* validated. The latter is only validated if we *assume* that the Henkin models considered are *restricted* to those that satisfy this schema (these are the *faithful* models). This means that Henkin models are *inadequate* to provide a semantics for second-order logic, given that soundness is a minimum condition for the adequacy of a semantics. If the semantics is not sound, it allows us to derive invalid consequences in the system—and *this* is surely unacceptable. However, it is *ad hoc* to impose these constraints on second-order models (namely, that the latter should be faithful Henkin models), since the reason why such constraints are assumed is simply to provide a semantics for second-order logic that is sound and complete. But as a result of these constraints, second-order logic also loses its capacity to characterize adequately theories with infinite domains, since Löwenheim-Skolem theorems then hold. In other words, it is clearly inadequate to introduce a semantics for second-order logic that makes this logic first-order (since we then lose the characteristic features of second-order logic). Moreover, no independent reason is provided as to why Henkin models should be invoked in the first place. And the only reason begs the question against the second-order theorist, since it assumes that completeness is a necessary condition for the adequacy of a semantics. Now, second-order *standard* semantics does not impose any such constraints; in this sense, it *is* adequate. Therefore, as opposed to Putnam's claim, there are arguments and criteria to choose between these semantics, and to prefer standard to Henkin models.

4 Ontological Problems: Second-Order Logic and Set Theory

I now consider problems for second-order logic on the ontological front. As is well known, Quine has famously criticized second-order logic for being nothing more than “set theory in sheep’s clothing” (Quine 1970, p. 66). Hence, once we allow for the quantification over predicate variables, we are committed to the whole set-theoretic hierarchy.

However, second-order logic is clearly weaker than set theory. Although the notion of second-order validity can be defined in set theory, one cannot define, in purely second-order terms, the notion of set-theoretical truth (Boolos 1975, pp. 518–519). Thus, just by using second-order logic, one cannot be charged with being committed to set theory.

Moreover, if we restrict ourselves to monadic second-order logic (which allows quantification only over monadic predicate variables), its ontological commitments do not go beyond those of first-order logic, given the introduction of plural quantifiers (see Boolos 1984, 1985; see also Simons 1982, 1997). The idea is that quantification over monadic predicates (such as ‘is a critic’) can be seen as a counterpart of a plural quantification in natural language. So, instead of understanding the monadic second-order existential quantifier, $\exists X$, as ‘there is a class’, according to Boolos (1984, p. 449), it can be read as the natural language plural quantifier ‘there are (objects)’. For instance, the well-known Geach-Kaplan sentence, ‘Some critics admire only one another’, despite not being equivalent to any first-order sentence,[7] can be straightforwardly symbolized in second-order logic.[8] However, in doing so we do not commit ourselves to the existence of additional items beyond those to which we are already committed. In order to understand this sentence (and, more generally, in giving a semantics for monadic second-order logic), we do not have to postulate, in addition to critics, a class of critics (or of whatever other objects we might be concerned with), and so our ontological commitments do not go beyond those of first-order logic.[9]

However, Resnik (1988) has criticized Boolos’s approach on the grounds that, in natural language, plural quantifiers are understood in terms of classes: whatever understanding we have of these quantifiers, it is articulated out of our understanding of set-theoretic notions. Thus, according to Resnik, we do not have a grasp of plural quantifiers independently of the notion of class, and hence cannot simply assume these quantifiers in the metalanguage of our semantics. As opposed to Boolos’s view, by using these quantifiers, we do not avoid ontological commitment to classes after all.

This criticism, however, is not decisive. Boolos may insist that we *do* have a grasp of plural quantifiers independently of sets, namely via plural forms in natural language. And even if, in natural language, there are several distinct plural forms,

[7] Kaplan’s proof of this fact is presented in Boolos (1984, pp. 432–433).

[8] Supposing that the domain of discourse consists of critics, and *Axy* means ‘*x* admires *y*’, the Geach-Kaplan sentence becomes: $\exists X\ (\exists x\ Xx \wedge \forall x\ \forall y\ ((Xx \wedge Axy) \rightarrow x \neq y \wedge Xy))$.

[9] Boolos’s idea is that the informal metalanguage in which we give the semantics for (monadic) second-order logic contains the plural quantifier ‘there are objects’, which is then used to interpret the second-order monadic quantifiers. [For further discussion, see Shapiro (1991, pp. 222–226), and Simons (1997)].

Boolos has only assumed plural *quantifiers*, which are reasonably well understood. In discussing this point, Shapiro has suggested that, at this stage, the debate between Resnik and Boolos enters a regress *ad infinitum*, since "Resnik might retort that even (our understanding of plural quantifiers) is mediated by set theory, *first-order* set theory, in which case we have indeed entered the regress […]" (Shapiro 1991, p. 226). But I think Boolos has still a point to make. Resnik's response disregards the fact that natural language (and, in particular, the plural forms it contains) has been articulated *prior* to set theory, and thus we cannot assume that our understanding of plurals (and, in particular, of plural quantification) depends upon set theory. Otherwise, Resnik would be committed to the counterintuitive claim that before the development of set theory we were unable to understand a sentence like 'Some critics admire only one another'. I do not deny, of course, that set theory may illuminate certain fragments of our language. I am only claiming that Boolos has a point in bringing plurals to bear on second-order quantification, since the former, at least in natural language, are well understood and have been previously introduced.

More generally, there is an issue here regarding what is meant, in Resnik's critique, by set-theoretic notions: (1) the notion of set could be taken in the intuitive sense, (2) it could be taken in a formal (axiomatic) sense, or (3) in a naive sense, as is used by mathematicians. Let me consider each of these interpretations of Resnik's point in turn.

It may be argued that, in providing an account of plurals, we need an *intuitive* notion of set. The idea is that the relation of the singular term to its plural form is the relation of the individual to the *class* of individuals of which it is a *member*. This membership relation is ultimately set-theoretical, and it is taken to be intuitive in the sense that no formal framework is assumed in order to spell it out.

This move is certainly reasonable and appealing. It is reasonable to invoke an intuitive notion of set, according to which to every property corresponds the set of objects that have this property. (This is, of course, a comprehension principle.) And this set corresponds to the plural of the term that describes the individuals that have the property in question. The problem faced by this view is that the intuitive notion of set is inconsistent. Russell's paradox has shown that this notion is unsustainable: without substantial constraints, the comprehension principle produces a contradiction.

Faced with the contradiction, two lines can be taken by the defender of the intuitive view. He or she can either adopt one of the extant axiomatic set theories, which provide the needed constraints on the comprehension principle, or he or she can adopt one of the several paraconsistent approaches to guarantee that, despite being inconsistent, the intuitive approach is not trivial.[10]

The problem with moving to axiomatic set theory is that we can no longer claim that we are dealing with an *intuitive* notion of set. As is well known, there are several non-equivalent set theories (Zermelo, Zermelo-Fraenkel, Kelley-Morse, von Neumann-Bernays-Gödel, Quine's NF etc.). Each of them adopts a different

[10] As opposed to classical logic, paraconsistent logic distinguishes the notions of inconsistency and triviality. A theory T is *inconsistent* if it has a theorem of the form ($A \wedge \neg A$), and T is *trivial* if all the formulas of the language are theorems of T. The threat posed by inconsistencies in a classical logic setting is the triviality they bring. In a paraconsistent framework, however, there are *inconsistent* but *non-trivial* theories (see da Costa 1974; da Costa et al. 2007).

strategy to avoid set theoretic paradoxes, while having enough resources to reconstruct classical mathematics. Now which of these theories, if any, correspond to the *intuitive* notion of set? I think the answer is *none*, since each of them incorporates moves that are far from being *intuitive*. For example, whereas the comprehension principle is certainly intuitive, the same is not the case of ZF's replacement. Quine's NF, for instance, depends upon the notion of stratification, which has no counterpart in the intuitive conception. And similar remarks apply to the other set theories. The existence of several non-equivalent set theories clearly shows that there is no unique way to move from an intuitive (inconsistent) conception of set to a (presumably consistent) axiomatic formulation of this notion. So, if Resnik claims that our understanding of plurals is ultimately dependent upon our (intuitive) understanding of sets, the latter cannot be spelled out axiomatically, since (1) axiomatic set theory has only been introduced recently in comparison with plural forms in natural language, and (2) there are *too many* set theories available. Which of them corresponds to the intuitive conception?

Note that without answering this question, Resnik cannot substantiate his claim that we have an intuitive understanding of sets that underlies our understanding of plurals. For given that there are non-equivalent set theories, there are theorems of one that are not theorems of the others. Therefore, by adopting one of them as the underlying account of sets, we would have different accounts of plurals. Moreover, it seems arbitrary to choose between these set theories on the basis of their characterization of plurals, given that they have not been proposed to accommodate this issue, but rather to provide frameworks for the formulation of mathematical theories.

It now becomes clear why Boolos does have a point in stressing that our understanding of plurals is not dependent upon sets. Terms for sets and plurals constitute different categories in our language, and if we try to understand the latter in terms of the former, we get involved in difficulties that have nothing to do with the proper understanding of plurals, but which concern mathematical issues. This is how it should be. After all, sets are properly mathematical entities, which are quite independent of linguistic forms in natural language.

Let us now consider the third alternative. When Resnik claimed that our account of plurals is articulated in terms of sets, perhaps what he had in mind was not axiomatic set theory, but *naive* set theory, which has been used for so long by mathematicians. According to this view, we need not provide an axiomatic treatment of set theory; we should only be careful in order not to consider sets that are "too big" [for a discussion, see Hallett (1984)].[11]

The problem with this suggestion is twofold. First, when exactly is a set "too big" to be admitted as a set? Any definite answer will have to spell out some *limitation of size* for admissible sets. But this is exactly what all classical set theories have done! Therefore, we would be back to their axiomatic treatment. Second, if we refuse to provide an account of size limitation, leaving the proviso about sets that

[11] I am making here a distinction between *naive* set theory and an *intuitive* notion of set on the grounds that the former is a *mathematical* theory, whereas the latter incorporates intuitive, pre-theoretic assumptions that we may have about sets, collections etc.

are too big to be sets just vaguely stated, we will be led to the difficulties that motivated axiomatic set theory in the first place: the inconsistencies and vagueness resulting from the naive conception. For if we are not careful about which sets to introduce—and it is only an axiomatic approach that tells us that—we may end up again with contradictions. Alternatively, if we do not adopt an axiomatic approach, we will leave set theory unavoidably vague, since the very notion of set is then left unspecified.

It does not help to claim that mathematicians have been using the naive account of sets without trouble, since (1) this is false, given the paradoxes, which clearly show that there is something wrong with this account (if we assume classical logic). Moreover, (2) if the mathematicians run into trouble, they simply stick to the most convenient *axiomatic* formulation of set theory in order to get rid of the difficulty. This explains why in fact naive set theory is so widely used, despite being inconsistent: the usual practice is to return to an axiomatic treatment if paradoxes emerge. But, as we saw, this provides no help for Resnik, since the adoption of naive set theory ultimately depends upon a return to axiomatic set theory, leaving the naive conception behind.

The only remaining alternative—if Resnik is to countenance a naive, inconsistent account of sets—is to change the underlying logic to a paraconsistent one. As we saw, in this setting, the inconsistencies are not threatening, since they do not lead to triviality. And this is certainly an alternative to take in the study of set theory (see da Costa 1986). The problem is that this is *not* an alternative open to Resnik, given his commitment to classical logic (see Resnik 1988, 1997). I conclude that the claim that our understanding of plurals depends upon sets is not tenable.

5 A Methodological Difficulty: The Adequacy of Second-Order Models

The fourth problem for second-order logic, posed by Melia (1995), moves the discussion to the issue of categoricity. According to Melia, Shapiro (1985) has not shown that the existence of unintended interpretations of first-order theories establishes the inadequacy of first-order languages for the formalization of mathematics:

> If the mere existence of many non-isomorphic interpretations of first-order mathematical theories shows that first-order languages are unable to characterize certain mathematical systems, then the mere existence of many interpretations of second-order theories seems equally well to show that second-order languages are unable to characterize the relevant mathematical systems. (Melia 1995, p. 129)

But, again, a reply can be presented. Given that all interpretations of a certain second-order theory (say, arithmetic) are *isomorphic*—as opposed to what happens in the first-order case—the relevant mathematical system *is* properly characterized: after all, there is only one type of structure which satisfies it. In particular, it does not matter that, given the many interpretations, different objects may be assigned to the same position in the structure. For example, in the case of arithmetic, any object can be assigned to zero, provided it plays the right function in the natural number structure. What is relevant is the *kind* of structure defined—and the categoricity result guarantees that only *one* type of structure is forthcoming. This is, of course, the sort of structuralist view that Shapiro favors (see Shapiro 1997). Therefore, *contra* Melia, there is a huge difference between the many *non*-isomorphic interpretations of first-order mathematical theories and the many *isomorphic* interpretations of second-order ones.

The only argument Melia puts forward against this structuralist move is the following:

> The structuralist cannot say that the powerset of [a given set] *d* is simply that set which contains all the subsets of *d* for, on his view, *any* object can be a subset of *d*, providing it plays the right kind of role in a system which instantiates the right structure. The question: 'how are two mathematicians to know that they have the same structure in mind?' is unanswered. Without an account of how mathematicians can communicate the structure of the powerset, structuralists have no reason to think that the second-order quantifier provides a solution to any part of the communication problem. (Melia 1995, p. 132)

The problem with this argument is that it seems to conflate the structure of a mathematical *system* (namely, set theory) with the structure of a mathematical *object* (namely, the powerset). From the fact that any object (which satisfies the appropriate conditions) can be a subset of *d*, it does not follow that two mathematicians cannot know what structure they are talking about when they discuss set theory. The question whether they have the same (set-theoretical) structure in mind is settled by establishing an appropriate isomorphism between the structures they are considering. And the existence of this isomorphism is compatible with there being different interpretations of the powerset. To communicate "the structure of the powerset" (say, by associating a given object to it) is entirely *irrelevant* for communicating the structure of set theory itself. This is because, according to structuralism, mathematical objects (such as the powerset) are at best positions in a structure—they lack any internal structure. One of the main points of structuralism is to argue that set theory (or any other mathematical theory for that matter) can be understood without any concern for the kind of objects that satisfy it.

Note also that, as a criticism of Shapiro's view, Melia's argument against structuralism begs the question. In Shapiro's hands, structuralism and the use of second-order logic are closely intertwined. As we saw, one of his main arguments for this logic derives from mathematical practice: since second-order languages allow categorical characterizations of theories with infinite domains, they are more adequate than their first-order counterparts to accommodate mathematical structures. When Melia argues that, due to the existence of different interpretations of second-order theories, they are not better than first-order ones, he assumes that a structuralist position in the philosophy of mathematics is not sustainable. For Melia assumes that different *isomorphic* interpretations (of second-order theories) are on a par with different *non*-isomorphic interpretations (of first-order theories). But it is crucial for the structuralist to distinguish between these two kinds of interpretation.

In order for the structuralist to stress the role of structure in mathematics, he or she needs a language that characterizes certain theories categorically, and a second-order logic is needed for that. So when Melia assumes that isomorphic and non-isomorphic interpretations are on a par, he is assuming that structuralism (with the use of a second-order language) is ultimately inadequate as a framework to formulate mathematical theories. After all, Melia is denying the importance of the distinction (between the two kinds of interpretation) upon which structuralism rests. But whether mathematical theories are adequately formulated by using second-order language is exactly the point at issue! Therefore, its denial cannot be assumed in a criticism of second-order logic.

To be fair to Melia, he does provide an argument against structuralism. The problem with this argument, as we have just seen, is its conflation of the structure of a mathematical object with the structure of a mathematical theory. In the end, the point does not go through.

The fifth problem for second-order logic was presented by Azzouni (1994). It is also a reaction against Shapiro's view, and similarly to Melia's problem, it is concerned with the import of categoricity results for second-order logic. I will discuss it in the next section.

6 An Epistemological Objection: Second-Order Logic and Referential Access to Mathematical Objects

6.1 How Not to Achieve Referential Access to Mathematical Objects

Azzouni is concerned with the problem of how we refer to mathematical objects, given that they are not subject to any causal constraints. He calls this problem the puzzle of referential access to mathematical objects (Azzouni 1994, p. 7). He then considers the claim that this problem can be solved by using second-order logic, since categorical definitions of certain mathematical structures become available in this context (Azzouni 1994, pp. 11–18). In resisting this claim, he has a clear target: Shapiro's view.

In his (1985) paper, Shapiro argues against the adequacy of certain logics for accommodating mathematical practice. We have seen some of his arguments against first-order logic. In order to reject some further alternatives, Shapiro argues that a logic cannot presuppose the very mathematical objects whose grasping and understanding we are trying to obtain. Thus, for instance, ω-logic (which is obtained from first-order logic by adding a new style of quantifier, ranging over natural numbers) cannot be taken as providing a solution to the issue of how we refer to natural numbers, since these numbers are presupposed in the formulation of this logic.[12]

According to Azzouni, the same sort of inadequacy is true of second-order logic. Indeed, in his view, "ascending to second-order logic cannot solve the puzzle of

[12] As Shapiro remarks: "the major shortcoming of ω-languages is that they assume or presuppose the natural numbers. Therefore, such a language cannot be used to show, illustrate, or characterize how the natural number structure is itself understood, grasped, or communicated" (Shapiro 1985, p. 733).

referential access, if it is not already solved by merely invoking the standard interpretation in the first-order case" (Azzouni 1994, p. 12). In order to argue for this point, he considers a non-standard model theory for first-order logic (a truncated model theory) that has the same models as second-order logic with standard semantics. And, he argues, since no one would accept that, by means of this truncated logic (as Azzouni calls it), we would have solved the problem of referential access, no one is entitled to conclude that second-order logic has solved it as well. But what is this truncated logic?

The idea is to restrict the class of models of the first-order predicate calculus. If S is a set of statements in the language of this calculus, and A is the class of models of S, a *truncation* of A is a class of models of S which does not include every model in A. A *truncated model theory* is then a class of models that does not include every model. Such a theory is constructed by requiring, for instance, that a predicate symbol in the language hold only of certain sets (say, two-membered sets) in any model (Azzouni 1994, pp. 12–13). The crucial feature, Azzouni stresses, is that first-order logic with a truncated model theory has greater expressive power than the first-order predicate calculus. However, we cannot claim to be able to solve the problem of referential access by adopting a truncation, after all:

> If I use a truncated model theory in which the natural numbers are 'truncated first-order definable', this certainly gives us no reason to believe that we have access to models of arithmetic that are isomorphic to the standard model—*for we have been given no explanation for how we are able to exclude the models not in the truncation*. (Azzouni 1994, p. 13; the italics are mine.)

Azzouni has to establish the equivalence between first-order logic with a truncated model theory and monadic second-order logic (Azzouni 1994, pp. 14–15). This is achieved by providing an appropriate method of translation between these logics. He first formulates the truncated model theory by admitting into the truncation only models M having the following properties: the domain of M is exhaustively and disjointedly divided into two classes, A and B, such that A is an arbitrary set and B is the powerset of A. Moreover, two logical constants, 'S' and 'E', are introduced, and their references are fixed so that, in every model, 'S' is assigned to B, and 'E' is assigned to the set $\{(a, b)\colon a \in A, b \in B, a \in b\}$. Finally, in every model, individual constants are assigned to A, and the extension of n-place predicates is a subset of A^n.

Azzouni then shows how to translate sentences of monadic second-order logic into the language of the truncated logic. For instance, the second-order induction schema for Peano's arithmetic

$$\forall X\,((X0 \wedge \forall x\,(Xx \rightarrow Xsx)) \rightarrow \forall x\,Xx)$$

becomes

$$\forall x\,(Sx \rightarrow ((E0x \wedge \forall y\,(Eyx \rightarrow Esyx)) \rightarrow \forall y\,Eyx)).$$

The idea is, of course, to consider this version of second-order logic as a two-sorted first-order logic, eliminating the additional quantifiers by relativizing them to predicates.

The last step is to provide a one–one mapping of monadic second-order logic onto the truncated model theory. Azzouni's definition is as follows (Azzouni 1994, p. 15). A model A_s of second-order model theory is mapped to a model A_t of truncated model theory in such a way that:

(1) If d_s is the domain of A_s, then d_t is the union of d_s and of the powerset of d_s.
(2) 'S' is mapped to the powerset of d_s.
(3) 'E' is mapped to the set $\{(a, b)$: $a \in d_s$, $b \in$ powerset of d_s, $a \in b\}$.
(4) Individual constants are mapped to the same items in d_t as they are mapped to in d_s. Moreover, n-place predicates are mapped to the same subset of d_t as they are in d_s.

As a result, Azzouni notes (*ibid.*, p. 15), the truncated logic has exactly the same metalogical properties as monadic second-order logic (with standard semantics). In particular, completeness, compactness and Löwenheim-Skolem do not hold; both logics have the same capacity to characterize infinite structures, and they have the same models. In this sense, they are equivalent.

The moral Azzouni draws from this equivalence is that there is no warrant to claim that second-order logic solves the accessibility problem, since it is "a simple act of fiat to fix the language and the model theory in the way I have to get […] truncated logic" (Azzouni 1994, p. 16). But then someone may wonder why, in the case of second-order logic with standard semantics, we do *not* end up with any impression of fiat. According to Azzouni, this is because "the acts of fiat are neatly tucked away in the second-order quantifier and the grammatical relation of predicate constant and variable to individual constant and variable" (*ibid.*, p. 17). Here is his argument:

> Normally, first-order logic is not taken to commit one to the extensions of the predicates as entities over and above what such extensions are composed of, but only to what the quantifiers range over. Thus the notation '*Pa*', which is a one-place predicate symbol syntactically concatenated with a constant symbol, is not taken to contain an (implicit) representation of ∈.
> However, as soon as we allow ourselves to quantify (standardly) into the predicate position, this is precisely how syntactic concatenation *must be* understood. Furthermore, and this is striking (or ought to be), syntactic concatenation in these contexts is *not* open to reinterpretation across models—it is an (implicit) logical constant.
> This explains, I think, […] why, when we make the concatenation involved here explicit in terms of 'E', it is clear that 'E' must be fixed in its interpretation across models (and one naturally observes that doing so presupposes, to some extent, our grasp of ∈) […]. (Azzouni 1994, p. 17)

Thus, in Azzouni's view, the second-order terminology and notation neatly masks the fact that, when using second-order logic, we are presupposing a grasp of certain mathematical notions (such as the membership relation of set theory). Hence, Shapiro has to reject this logic on the same grounds as those he gives in rejecting ω-logic, and withdraw his claim that second-order logic offers a solution to the problem of referential access.

But how compelling are Azzouni's arguments?

6.2 Achieving Referential Access to Mathematical Objects

In my view, the arguments Azzouni put forward are far from being decisive. First, note that the truncation was obtained, in a clearly *ad hoc* way, *assuming the information provided by second-order logic* about which models to exclude from the class of first-order models. It is because we have second-order logic at our disposal that we are able to construct the equivalent truncated logic (the former provides heuristic guidelines for the construction of the latter). So, Azzouni can only maintain the adequacy of his construction if he *assumes that second-order logic solves the problem of referential access* in the first place. But if he grants this point (as he should, in order to have his case), the second-order theorist can simply reject the truncation on the basis of its *adhocness*. Far from establishing the inadequacy of second-order logic to accommodate the referential access, the truncation spells out the advantage of this logic, since it allows us to clearly distinguish between an *ad hoc* construction (which assumes second-order logic, but only mimics its features) and an adequate, original one (the class of categorical theories achieved through the proper resources of the second-order language).

Second, Azzouni's explanation of why we do not have the impression of fiat when we use second-order logic misunderstands the nature of second-order quantifiers. His claim that "as soon as we allow ourselves to quantify (standardly) into the predicate position [...] syntactic concatenation *must be* understood" as "an (implicit) representation of $\in$" simply does not hold. As we saw, Boolos introduced plural quantification precisely to avoid commitment to classes in interpreting (monadic) second-order quantifiers. In fact, he has shown (in Shapiro's words), "how a single predicate *R* (in the metalanguage) can code an assignment of 'values' to the second-order variables of the formal language" (Shapiro 1991, pp. 223–224). The idea is that $\langle V, v \rangle$ is in *R* iff *V* is a second-order variable and *v* stands for the objects to be assigned to *V*. According to Boolos, a second-order formula of the form *Vv* is then interpreted thus: we say that *R* and an assignment *s* to the first-order variables (only) satisfy *Vv* iff $R\langle V, s(v)\rangle$ [where '$\langle , \rangle$' is the ordered-pair function sign; see Boolos (1985), p. 336]. Intuitively, we say that *R* is the *plural* relation ('is the plural of'), and that '$R\langle V, s(v)\rangle$' is the case if the object $s(v)$ is *one of them* (i.e. one of the *V*'s). Thus, there is no need for understanding concatenation in terms of set-theoretical membership. Moreover, any impression of fiat can be explained not set-theoretically, but by an appropriate understanding of second-order quantifiers, such as that provided by Boolos's approach via plural forms in natural language.

Moreover, Azzouni's remark that "syntactic concatenation in these [second-order] contexts is *not* open to reinterpretation across models—it is an (implicit) logical constant" can be explained straightforwardly, in Boolos's approach, with *no* commitment to $\in$. The idea is simply that, as we saw in the previous paragraph, syntactic concatenation is accommodated by the ordered-pair function. But since this function has fixed properties, which do not change across models, it does not allow syntactic concatenation to be open to reinterpretation.

Now, Azzouni's trick in the construction of the truncated model theory is to select those models such that '*E*' is interpreted as '$\in$'. As a result, sentences of the form *Xx* become $x \in X$, which means that second-order predication is interpreted as membership. This move can be traced back to Quine's (1970) criticism of second-order logic. Quine's advice to the logician was to avoid second-order quantification and adopt a formulation in terms of membership; that is, instead of writing '$\exists X\, Xb$', he or she should write '$\exists \alpha\, b \in \alpha$'. Now, the problem with this suggestion, as Boolos has stressed (1975, p. 40), is that it is neither validity-preserving nor implication-preserving. For instance, although '$\exists X\, \forall x\, Xx$' is valid, the same is not the case for '$\exists \alpha\, \forall x\, x \in \alpha$'. Moreover, although '$x = z$' follows from '$\forall Y\, (Yx \leftrightarrow Yz)$' by logic alone, it does *not* follow from '$\forall \alpha\, (x \in \alpha \leftrightarrow z \in \alpha)$' only by logic, but it requires some set theory. It now becomes clear why second-order quantification and set-theoretical membership are indeed quite distinct. And with this distinction in hand, we can reject Azzouni's insistence that second-order logic assumes set theory.

Furthermore, notice that, in Azzouni's construction of the truncated model theory, the introduction of the two predicates, '*E*' and '*S*', has the role of 'extending' the first-order quantifiers in such a way that we achieve the expressive power of second-order logic *without* actually using second-order quantification. Now, in the standard semantics for second-order logic, there are no restrictions on the sets one considers when interpreting the range of second-order variables. And this is crucial if we are to make second-order logic *really* second-order. In order to make *first*-order logic resemble second-order, Azzouni has to introduce '*E*' and '*S*', which are to be read, respectively as 'is a member of' and 'is a set'. In other words, in order to achieve the expressive power of second-order languages in a first-order setting, Azzouni *has to introduce set-theoretic notions*. But *without any recourse to these notions*, second-order logic is able to achieve its expressive power. So the most we can say Azzouni has established is that, without set-theoretic talk, *first*-order logic lacks the expressive power of second-order logic. Would this prove that the latter already *presupposes* set theory? Surely not! As we saw, the *point* of Boolos's plural interpretation of second-order quantifiers is precisely to establish that we are not committed to set-theoretic talk in order to provide a semantics for (monadic) second-order logic.

7 Conclusion

As we have just seen, Azzouni has not established the inadequacy of second-order logic to solve the problem of referential access.[13] Furthermore, each of the problems posed by Putnam, Resnik, Quine, and Melia can also be dealt with. Clearly, this is not to say that second-order logic is unshakeable. It may well be that further problems will appear. But I hope I have established that this logic is far more robust than the above critics have supposed.

[13] Of course, this does not establish the truth of Platonism, given that we can refer perfectly well to things that do not exist (see Azzouni 2004; Bueno 2005).

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