Relativism in Set Theory and Mathematics

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Abstract

According to the mathematical relativist, mathematical statements, if true, are relatively true only—that is, they are true relative to (i) the specification of concepts and theories that characterize the relevant mathematical domain, (ii) the mathematical framework in which such mathematical truths are formulated, and (iii) the logic that is employed in the derivation of such truths. This chapter surveys mathematical relativism and its significance. First, it clarifies what mathematical relativism is not. It then examines a formulation of the view in which conceptual, structural, and logical relativity play a central role, but which still preserves the objectivity of mathematics. The implications of mathematical relativism for the ontology of mathematics are then considered. While some forms of Platonism and nominalism in mathematics are compatible with mathematical relativism, others are not. Finally, some remarks about the significance of mathematical relativism conclude the chapter.

Introduction

Mathematical relativism is the view according to which every mathematical truth is relatively true only. This formulation of the view extends to mathematics the consistent characterization of perspectival relativism that Steven Hales (2006) has defended. So there are no true simpliciter mathematical statements. Truth in mathematics is relative. But to what is it relative?

To answer this question is to give some content to the relativist claim. Briefly put, mathematical truths are relative to (a) the concepts that characterize the relevant mathematical field or the particular mathematical theory under consideration, (b) the mathematical framework (e.g., a given set theory) in which such truths are formulated, and (c) the logic that is used to derive the statements. We have then at least three distinctive sources of relativity: conceptual, structural, and logical. Although distinct,
these sources of relativity are obviously connected, given that there are dependence relations among them.

For example, typically, the set theory one considers depends on the underlying logic that is adopted. The Russell set (the set of all sets that are not members of themselves) cannot be formulated in the usual set theories, such as Zermelo–Fraenkel set theory, which are constructed using classical logic and aim to be consistent. However, the Russell set can be straightforwardly expressed in a paraconsistent set theory, that is, a set theory that is constructed using a paraconsistent logic. Such a logic is one in which not everything follows from a contradiction (see da Costa et al. 2007). In classical logic, if a theory T is inconsistent — i.e. if a statement of the form $\neg A$ and $\neg \neg A$ is a theorem of $T$ — then $T$ is trivial, since every statement in $T$’s language is then a theorem. Given paraconsistent logic, however, we can distinguish inconsistency from triviality. In particular, there are inconsistent theories that are non-trivial, such as a paraconsistent set theory in which the Russell set is constructed, but not everything follows from that theory (see da Costa et al. 2007).

This example illustrates some relations between conceptual, structural, and logical relativity. The statement “the Russell set exists,” if true, is relatively true only: it is true in a paraconsistent set theory, but false in Zermelo–Fraenkel set theory (assuming that the latter is consistent). Clearly, the truth of that statement depends on the logic that is used: a paraconsistent logic, rather than classical logic, needs to be in place if the statement is to be true without triviality (logical relativity). This, in turn, requires a change in the set theory under consideration: from classical to paraconsistent set theory (structural relativity). And with that change a different conception of set, which allows for the possibility of inconsistent sets without triviality, is then invoked (conceptual relativity).

In this chapter, I will discuss the significance of these three components to mathematical relativity, dispelling along the way some misunderstandings that this view can provoke. I will start by addressing what mathematical relativity is not, indicating later what it is. To be clear about what is at issue, it is important to identify some features that mathematical relativity does not have, and also some features that it does.

1. Mathematical Relativism: Does Everything Go In Mathematics?

Relativism is sometimes presented as the view according to which anything goes. Paul Feyerabend (1988: 14–19), rather provocatively and somewhat jokingly, suggests that the only principle that does not inhibit scientific progress is: anything goes. His own discussion of relativism is more nuanced though (1987: 19–89). He relies on the trouble-making practice of ancient Greek skeptics, such as Sextus Empiricus (1994), who argued that those who make claims about the ultimate nature of reality are often unable to establish these claims according to their own standards. This is the sort of practice that relativists explore. By calling on this practice, it becomes increasingly clear that no single method is adequate to every domain of inquiry, that a plurality of approaches is preferable. But we are still miles away from establishing the point that anything goes.
Whatever the fate of relativism in the context of scientific inquiry, mathematical relativism is not the claim that anything goes in mathematics. Mathematical relativism had better not be stated in this way — or to entail anything along these lines. After all, given that mathematics is a fairly constrained practice, any view about mathematics with these features would be false as an account of mathematics in general and of mathematical activity in particular. Such a wild formulation of relativism about mathematics is both descriptively inadequate and normatively empty. It is descriptively inadequate given that mathematicians do meet various constraints while doing mathematics: they adopt an underlying logic (even if only implicitly); they embrace standards of rigor (even if the latter change in time); they work within certain frameworks, which constrain the acceptable definitions, allowable moves in a proof, and specify the suitable language they can use. Given these constraints, it is simply not the case — as a description of actual mathematical practice — that everything goes.

Similarly, it is normatively empty to state that anything should go in mathematics. The constraints mentioned above not only have informed mathematical practice over the centuries, but they also should have informed that practice. Without constraints of this sort, we would not have mathematics, as we know it: we would have an entirely different body of work, and a dramatically different enterprise. It simply would not be mathematics — certainly not as we recognize it now.

As will become clear, everything in mathematics is in principle revisable. Definitions of mathematical concepts, fundamental mathematical principles, even the underlying logic are all revisable, and have been revisied, in mathematical practice. This does not mean, however, that these concepts, principles, and logic can all be revised at the same time. Typically, changes in mathematics take place gradually. Even though, over time, there may be significant differences between distinct practices in mathematics, given changes in their definitions, results, and even their logic, the process of transformation evolved in gradual steps, with one change here (say, the refinement of a given concept), a revision there (a new theorem that could be obtained given the revised concepts), and a more unusual change over there (e.g., with a new underlying logic being adopted). In this way, mathematical change is indeed a gradual process. It takes place at the levels of theorems (which include conceptual refinements introduced by new definitions), methods (which include particular standards of rigor and the underlying logic), as well as goals (which include certain values, such as informativeness, elegance, and consistency). But these changes, although individually all possible, do not happen all at once. In this respect, the situation is analogous to what happens in the empirical science — as Larry Laudan (1984) has described via the reticulated model. As a result, even from such a methodological perspective, not everything goes in mathematics. Mathematical relativism should then be emphatically distinguished from any form of mathematical trivialism (the claims that everything is derivable in mathematics, or that anything goes in mathematics).

2. Conceptual, Structural and Logical Relativity in Mathematics

Mathematical truths are often considered to be necessarily true. On this traditional view, mathematical truths are then absolute, in the sense that they are not relative to
some context or some other pragmatic feature. Despite being widely held, it is not clear that this view is correct. According to the mathematical relativist, everything mathematically true is only relatively true, i.e. true in some perspective. If this view is right, the absolutism of the traditional conception is inadequate. Let us see why.

Crucial for the mathematical relativist, just as for the perspectival relativist (Hales 2006), is the concept of a perspective. What are perspectives in the mathematical case? Perspectives are the particular mathematical principles that characterize and define the meaning of the terms used in a certain domain of mathematics. Since, as we will see, the truth of mathematical statements is relative to such principles and the concepts that are invoked in the latter, a form of conceptual relativity emerges. But the mathematical statements in question may be formulated in different overall frameworks (such as different set theories, different category theories, or some other foundational framework). Since, as we will see, the truth of mathematical statements is relative to the particular framework in which such statements are expressed, a form of structural relativity results. Given that such foundational frameworks are mathematical theories in their own right, structural relativity is a form of conceptual relativity. However, because of the significance that the choice of a given foundational framework has to mathematical practice, it is worth considering structural relativity separately. Moreover, perspectives in mathematics also depend on the underlying logic that is used to formulate the relevant mathematical theories. Since, as we saw, the truth of mathematical statements is relative to the underlying logic of the mathematical theories in which such statements are formulated, a form of logical relativity emerges. Logical relativity, as noted above, clearly impacts on conceptual and structural relativity. I will illustrate these points in turn.

In order for us to consider whether it is true that sets exist, we need first to specify which sets are under consideration. The term “set” is referentially indeterminate, and we need to disambiguate between various different extensions of this term: each set theory specifies a particular extension, a possible way of specifying the meaning of the term “set,” and the corresponding properties that such sets have. Each set theory provides a perspective in which we can assess whether certain claims about sets are true. The specification is done via suitable comprehension principles for sets. Interestingly enough, different set theories do that differently, and as a result, what is true in a set theory is so only relatively – relatively to the perspective that is adopted. Consider, for example, the differences between the perspectives offered by the axioms for Zermelo-Fraenkel (ZF) and von Neumann–Bernays–Gödel (NBG) set theories. It is true in the latter perspective, but false in the former, that there are proper classes (totalities that are too big to be sets). It is false in the perspective of ZF, but true in ZFC (Zermelo-Fraenkel with the axiom of choice), that every set can be well ordered. It is true in the perspective of NBG, but false in ZFC, that set theory can be finitely axiomatized in a first-order language. Even the claim that sets exist is not absolutely true, since there are formulations of second-order mereology that, coupled with some assumptions regarding the existence of enough mereological atoms, are provably as powerful as certain set theories, but which nowhere invoke sets (Lewis 1991).

The description so far, although focused on set theory, can be just as easily applied to different branches of mathematics, such as analysis, geometry, or arithmetic. In all these cases, one needs to determine the objects of investigation by specifying suitable
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perspective in which the truth of mathematical statements is assessed.

In the case of analysis, there are many different options, ranging from classical,
standard analysis (the usual analysis done in second-order logic) through classical
nonstandard analysis (such as the systems articulated by Robinson 1974) to non-
classical standard analysis (when standard analysis is developed using some non-
classical logic). In each case, suitable comprehension principles need to be specified and
explored, and, without such principles, it is simply not determined which objects we
are talking about. Moreover, there are some statements true in the perspective of non-
standard analysis that cannot even be expressed, let alone be true, in standard analysis
(Henson and Keisler 1986). Similarly, there are results about the calculus that are true
in the perspective of non-classical, inconsistent analysis — when standard analysis is
formulated with a paraconsistent logic — that are false in classical standard analysis
(Mortensen 1995). Just as with what happens in set theory, mathematical truths in
analysis are only relatively true.

Similarly, there are very different geometries, from classical Euclidean geometry
to various non-Euclidean geometries. What is geometrically true is only relatively
true — relatively to the perspective one takes. For instance, the fifth postulate of
Euclidean geometry is true relatively to that geometry; false in non-Euclidean
gometries. And once again, if we change the underlying logic, different perspectives
emerge, and with them, we obtain still more variation of truth-values of geometrical
propositions.

Even in the case of arithmetic, we need to specify the perspective we are taking, by
invoking suitable comprehension principles. Suppose we want to determine what is the
nature of natural numbers; that is, what kind of objects they are. Depending on the
perspective that we adopt, we obtain radically different answers: we can adopt a neo-
Fregean construction of such numbers, or one of many possible reformulations of
natural numbers in set theory, or the traditional formulation of arithmetic following
the Peano axioms. In each case, we obtain a different system, and a very different
answer to the question of the nature of natural numbers: Fregean objects, various kinds
of sets, or whatever satisfies the Peano axioms. For convenience, we can simply stipu-
late that all such formulations are adequate, and show that they are equivalent for
certain purposes (basically, the same results about numbers can be obtained in each
system). But this does not change the fact that, strictly speaking, each particular
formulation of arithmetic provides a different characterization of the nature of numbers.
But other perspectives can be adopted: if we change the underlying logic to a paracon-
sistent one, there will be arithmetical propositions that are true in that perspective but
which are false in classical accounts. Or if we adopt the perspective of arithmetic for-
mulated in a second-order logic with standard semantics, it is false that there are non-
standard models of arithmetic. But this claim is true in the perspective of first-order
arithmetic. Once again, arithmetical truth is relative true only.

For these reasons, it is crucial to be explicit about the perspective that is adopted
when we assess the truth of mathematical statements. We need to be explicit about
which comprehension principles are introduced in a particular branch of mathematics.
As one would expect in a relativist account, the truth of mathematical statements is
now tied to particular perspectives — the particular comprehension principles that
specify the meaning of the mathematical terms employed in a given context. (The context here is determined by the specification of the comprehension principles in question.) So, the truth of mathematical statements is always dependent on the perspective that is adopted, the comprehension principles that are invoked. Such principles determine a particular, internal perspective in terms of which the relevant mathematical statements are assessed.

Given mathematical relativism, no claim is made about the truth of mathematical statements beyond the context determined by the comprehension principles in question; that is, outside the framework of perspectives. Within a given perspective, it is true (constitutively true) that there are mathematical objects of the appropriate sort. But nothing is claimed (or can be claimed) beyond such individual perspectives. After all, outside the perspectives determined by such principles, it is not clear what is meant by the mathematical terms under consideration: one needs comprehension principles to specify their meaning. As a result, since the truth of mathematical claims changes across perspectives, mathematical statements are only relatively true. In the end, according to the mathematical relativist, mathematics is not absolutely true.

As noted above, mathematical relativism does not lead to the conclusion that everything goes. In fact, mathematical relativism is a perfectly reasonable view. To claim that everything mathematically true is relatively true only is not to claim that everything is true. So, mathematical relativism is not a form of trivialism. There are things about mathematics that the mathematical relativist can say are untrue, by identifying perspectives in which what is said is untrue. And the mathematical relativist can identify many things as being false. For instance, to say that in classical arithmetic there aren’t infinitely many prime numbers is to say something untrue. The mathematical relativist would happily concur.

Note also that the mathematical relativist need not be committed to the claim that every perspective is equally acceptable. There are perspectives that are more fruitful than others, richer in the implications they have, and in the problems they raise. Such perspectives, the mathematical relativist can insist, are to be preferred and explored further. In fact, one way of making sense of the debate between set-theoretic and category-theoretic foundations of mathematics (see, e.g., Hellman 2003) is to frame it as an issue of structural relativity. Depending on the particular perspective that is adopted—say, a given set theory or a particular category theory—distinct ways of reconstructing mathematics emerge. Certain constructions are more natural in one framework than in the other, and vice versa, and the choice between such frameworks is ultimately made on the basis of pragmatic considerations.

The same points can be extended to logical truths. Such truths are also relatively true only—true in some perspectives, and false in others. What are such perspectives? In the case of logic, perspectives are the domains of discourse to which logics are applied. Different domains are modeled by different logics. Inconsistent domains are modeled by various paraconsistent logics, incomplete domains by various constructive logics, quantum domains by quantum logics, and so on. What is logically true is then relatively true: true in some perspectives (in some domains), but false in others. For example, explosion—the principle according to which everything follows from a contradiction—is true in the consistent perspective governed by classical logic, but false in the inconsistent perspective of paraconsistent logic. Mathematical relativism leads
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ows from a non-constructive logic, but false relativism leads to a very natural form of logical pluralism (Beall and Restall 2006; Bueno and Shalkowski 2009).

Although the underlying logic of typical mathematical theories is classical, this is not always the case, given the formulation of various mathematical theories via non-classical logics. For example, analysis has received an elegant and insightful treatment via intuitionist logic (Bishop and Bridges 1985); various inconsistent mathematical theories have been formulated via paraconsistent logic (Mortensen 1995); and quasi-set theories suitable for the interpretation of the foundations of quantum mechanics have been developed using Schrödinger logic (French and Krause, 2006). This is just a small sample of the plurality of logics that have been used in mathematics. They offer additional support to the claim that mathematical truths are relatively true only. The perspectives involved in these cases crucially rely on the non-classical logics that are used, but also on the concepts and the overall framework in question. In the end, conceptual, structural, and logical relativity all go hand in hand.

3. Mathematical Relativism and Mathematical Objectivity

It may be thought that mathematical relativism is incompatible with an objectivist view of mathematics, that is, with the view according to which the truth of mathematical statements is an objective matter. But this is not correct. For the objectivist, we cannot simply decide or stipulate what the truth-value of mathematical statements is. Even the truth-value of mathematical axioms is not simply stipulated, given that which axioms are adopted depends on how fruitful and mathematically interesting the resulting system turns out to be. In this sense, truth in mathematics is not of our own making.

The truth-value of mathematical statements depends on the particular concepts and mathematical principles that characterize a certain domain of mathematics as well as the underlying logic that is used (more or less explicitly) to obtain the relevant results. Once these three components are in place, it is not up to us to decide what the truth-values of mathematical statements ultimately are. This does not entail, of course, that we know what are these truth-values. For example, we may not have been able, so far, to either derive certain mathematical statements from the relevant principles, or to construct a counter-example to such statements, or to show that the statements in question are independent of the principles. In this case, clearly we do not know, at least at the moment, the truth-value of the statements in question.

If the mathematical statements turn out to be independent of the relevant principles (as the axiom of choice and the continuum hypothesis are from Zermelo–Fraenkel set theory), then the principles clearly do not determine the truth-value of the statements under consideration. Perhaps in this case one may think that we get a bit closer to being able more or less to stipulate the truth-value of the statements in question. However, even in this case, there are constraints imposed by the consistency with other assumptions made by extensions of the relevant theory as well as by related theories (some of which may be able to prove the statements in question). So, even in this scenario, there is still objectivity.

Preserving the objectivity of mathematics is an important requirement for any account of mathematical practice. After all, as noted above, that practice is significantly
constrained by shared assumptions that characterize a certain domain of mathematics (such as the axioms for a group structure, or the axioms of Zermelo-Fraenkel set theory). Mathematical practice is also constrained by inferential techniques that are used to derive the relevant results (such as various methods for constructing countable structures in model theory). Once these assumptions and techniques—which include a given logic—are accepted, it is an objective matter whether some result is obtained or not.

In order to characterize such objectivity, do we need the commitment to the existence of mathematical objects, relations, or structures? Platonists about mathematics insist that such objects, relations, and structures exist and are abstract (in the sense that they are causally inert and are not located in space-time). Platonists often emphasize that it is because mathematical objects, relations, and structures exist that mathematics is ultimately objective. As a description of independently existing mathematical objects, relations, and structures, mathematical theories turn out to be true (if the relevant objects, structures, and relations are correctly described) or false (otherwise). The objectivity of mathematics is then grounded on the existence of the relevant mathematical objects (as well as structures and relations).

This move, however, does not go through. It is unclear that the existence of mathematical objects, relations, and structures does any work to support the objectivity of mathematics. After all, if mathematical objects, relations, and structures turn out not to exist, it is unclear that anything would change in mathematical practice (Azzouni 1994). Mathematicians would continue to do their work in precisely the same way as they currently do: proposing, articulating, and refining mathematical definitions and principles, and drawing consequences from them. The actual existence of mathematical objects is largely irrelevant for that. Clearly, in this case, existentially quantified mathematical statements (such as, “there are infinitely many prime numbers”) will not be true if they are interpreted as being about independently existing mathematical objects, given that, by hypothesis, such objects do not exist. However, such statements would be true if interpreted as being about the objects introduced by the relevant mathematical principles. In this case, we have a situation analogous to the attribution of truth in a fiction. Although “Sherlock Holmes was a detective” is not taken to be true as a description of the actual world, the assessment changes entirely if we make clear that the situation we are describing is within a fiction (in this case, the Holmes stories). After all, “According to the Holmes stories, Sherlock Holmes was a detective” is clearly true. Similarly, in the case of mathematics, “There are infinitely many prime numbers” is not true if numbers do not exist. However, the assessment changes dramatically if the situation we describe is restricted to the fiction of arithmetic. In this case, “According to arithmetic, there are infinitely many prime numbers” is true (Field 1989).

This form of nominalism, however, that explicitly introduces a fiction operator ("According to fiction F, ...") does not seem to be adequate. After all, it changes the syntax of mathematical statements to preserve verbal agreement with Platonism. It would be better not to make such changes, while still preserving the lack of ontological commitment to mathematical objects, structures, and relations. This is possible if we note that when mathematicians introduce concepts via suitable comprehension principles, nothing more is needed to refer to the relevant mathematical objects. However, no commitment to the existence of such objects is thereby made, given that
domain of mathematics.

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Commitment to the existence of mathematical objects, relations, and structures turn out to exist
independent of how they are described by the relevant set-theoretical principles,
methods, and techniques, given that the existence of these objects, structures, and relations
plays no role in actual practice, the objectivity in mathematics needs to be obtained
by some other means. Clearly, mathematicians do have access to mathematical principles,
methods, and techniques; these are the sorts of things that can be invoked to obtain objectivity. But an account of mathematical objectivity that is based on the
access to such principles, techniques, and methods does not rely on Platonism. In fact,
it relies on the non-Platonist account of objectivity suggested above.

4. Mathematical Relativism and the Ontology of Mathematics: Platonism

If the mathematical relativist can obtain the objectivity of mathematics without a
Platonist ontology, the question arises as to whether Platonism is even compatible with
mathematical relativism. It may seem that it is not. It turns out, however, that the situation
is more complex. As will be argued now, mathematical relativism is compatible with
certain forms of Platonism and incompatible with others. (In the next section, I
will raise the corresponding issue for nominalism.)

Platonism about mathematics, as noted, is the view according to which (a) mathematical
objects, relations, and structures exist — independently of our linguistic practices and psychological processes. Moreover, (b) mathematical objects, relations, and structures are abstract, in the sense that they are causally inert, and are not located in
space-time. An important feature of mathematical practice, according to the Platonist,
is that mathematicians are supposed to identify, characterize, and describe such mathematical objects, relations, and structures.

There are many versions of Platonism, depending on the way in which the relevant
mathematical objects, relations, or structures are characterized, and depending on
which of these items is taken to be ontologically fundamental. According to the Fregean
Platonist, mathematical objects (with the possible exception of geometrical objects) are
ultimately logical constructs, obtained by logic and definitions. For instance, given that
no objects fail to be identical to themselves, 0 can be characterized as the number of
objects that are not identical to themselves. 1 is the number of objects that are
identical to 0 (there is only one such object), 2 is the number of objects that are
identical to either 0 or 1 (there are exactly two such objects), and so on. Arithmetic
can then be developed roughly along these lines, just in terms of logic and definitions
(see, e.g., Frege 1974; Hale and Wright 2001; Boolos 1998).

On the Fregean picture, there is a fixed totality of abstract objects that mathematical
theories describe. The assumption is also made that there is a single logic that is used
in the characterization and description of such objects. Given that the mathematical
relativist emphasizes the plurality of mathematical structures, their dependence on the
particular concepts and theories that are invoked in the characterization of the relevant
objects, and the possibility of using different logics in such characterizations, it seems that mathematical relativism is incompatible with Fregean Platonism. The Fregean Platonist will, no doubt, conclude: so much the worse for mathematical relativism! The relativist will return the favor, challenging the adequacy of the Fregean picture in light of the plurality and variety of contemporary mathematics. Without substantial revisions, it is unclear that the Fregean Platonist is in a position to make sense of such plurality.

According to Gödelian Platonism, we have access to the mathematical realm—in particular to the realm of sets—in a way similar to our perceptual access to the physical world. We have, Gödel claims, "something like a perception of the objects of set theory" (1964: 485). That is, we are able to "perceive" these objects as having certain properties and lacking others similarly to the way we perceive physical objects in our environment. That we have such a perception of set-theoretic objects is supposed to be "seen from the fact that the axioms [of set theory] force themselves upon us as being true" (Gödel 1964: 485). And just as the perception of physical objects is fallible, so is the "perception" of sets. We may think that we see a colorful bird near a tree, just to realize, when we get close enough, that we have been looking at a bright piece of paper. Similarly, we may think that we are "perceiving" a given set when, in fact, we are "perceiving" a different object. However, in both the physical and the mathematical cases, the relevant perception is robust: we may be mistaken about the particular content of the perception, but we cannot be mistaken that we perceive something (leaving cases of hallucination aside). Moreover, similarly to Frege's picture, underlying the Gödelian proposal there is the assumption of uniqueness: the mathematical realm—in particular, the set-theoretic reality—is unique, in the sense that there is fundamentally one mathematical domain that we try to characterize and describe with our mathematical theories.

Given such uniqueness assumption, Gödelian Platonism is incompatible with mathematical relativism. The relativist notes the plurality of mathematical structures, relations, and objects, as well as of underlying logics that can be used in their characterization, and this immediately conflicts with the uniqueness requirement introduced by the Gödelian Platonist. Of course, similarly to the Fregean, the Gödelian Platonist may then insist: so much the worse for mathematical relativism! But in this case, the Gödelian Platonist owes us an account of the status of the plurality of mathematical concepts, structures, theories, and logics discussed above. Without such an account, it is difficult to understand how the Gödelian Platonist can make sense of such a plurality.

According to Full-Blooded Platonism, all mathematical objects that logically possibly exist actually do exist (Balaguer 1998: 53). The notion of possibility is here taken in its broadest sense, namely, as logical possibility (Balaguer 1998: 5–6, 69–73). In other words, the picture of mathematical reality that emerges is one of plenitude: all logically possible mathematical objects exist. From non-Cantorian sets to non-separable Hilbert spaces, from unusual metric spaces to yet unknown solutions to weird differential equations, every (consistent) mathematical object is welcome to this truly overpopulous mathematical paradise. Unlike both Fregean and Gödelian Platonism, Full-Blooded Platonism has no uniqueness requirement as part of the ontology of mathematics.
By rejecting the uniqueness of the mathematical realm— and hence by denying that there exists a single, well-structured domain of mathematical objects that the relevant mathematical theories describe—the Full-Blooded Platonist advances a view that is compatible with mathematical relativism. This is the case particularly if the Full-Blooded Platonist also acknowledges the role played by different logics in the formulation of mathematics. The Full-Blooded Platonist may insist that, ultimately, mathematicians aim to describe a certain robust, stable, standard part of the mathematical realm by trying to characterize a standard conception of various mathematical fields, such as arithmetic and analysis. But the point still stands that, on the Full-Blooded Platonist picture, there is a plurality of additional parts of the mathematical realm that need not be successfully picked out by the standard conception. By rejecting the uniqueness requirement, the Full-Blooded Platonist offers a formulation of Platonism that is compatible with mathematical relativism.

For these reasons, mathematical relativism is independent of Platonism. Although certain Platonist views are incompatible with mathematical relativism, such as Fregian and Gödelian conceptions, other versions of Platonism, such as Full-Blooded Platonism, are compatible with it. In the end, mathematical relativism does not settle the issue of the existence of mathematical objects.

5. Mathematical Relativism and the Ontology of Mathematics: Nominalism

How about nominalism—is it compatible with mathematical relativism? The answer, once again, is: it depends. Some versions are compatible; some are not. Similarly to Platonism, there are different formulations of nominalism about mathematics. According to the nominalist, mathematical objects do not exist or, at least, need not be taken to exist in order to make sense of mathematics. I will discuss some versions of this view, and their connections to mathematical relativism.

According to conservative nominalism, mathematical theories need not be true to be good. They need to be conservative, that is, consistent with every consistent claim about the physical world (Field 1980; 1989). We say that a claim is nominalistic if it contains no mathematical vocabulary, and does not refer to mathematical objects. If a mathematical theory M is conservative, and if we apply a mathematical theory to a body of nominalistic claims N, then every nominalist conclusion that can be derived from N + M can be derived from N alone. In this sense, mathematical theories, if conservative, are ultimately dispensable—as long as we have the relevant body of nominalistic claims to apply the mathematical theories to. Field’s program aims to provide such a body of claims, and he has reformulated Newtonian gravitational theory without quantification over mathematical objects (Field 1980). But it remains unclear whether such reformulations can be offered for quantum mechanics and other central scientific theories (Bueno 2003).

The issue, however, is not whether Field’s program is viable or not, but rather whether this version of nominalism is threatened by mathematical relativism. It may seem that it is not. After all, if mathematical objects do not exist, there simply is no fact of the matter as to whether mathematical theories, if true, are relatively true only. Furthermore, since,
for the conservative nominalist, truth is not even the appropriate norm for mathematics, the relativity issue does not even emerge. Is this assessment correct?

If we assume, with the conservative nominalist, the non-existence of mathematical objects, existentially quantified mathematical theories would not be true. In particular, as noted above, any existentially quantified theorem—such as “there are infinitely many prime numbers”—would be false, since no such numbers exist. As a result, the statement of mathematical relativism—namely, mathematical statements, if true, are relatively true only—would be vacuously satisfied, assuming that a material conditional is used in such statement. How can the conservative nominalist reject mathematical relativism then? He or she would need to provide true mathematical statements that are not relatively true. But this is precisely what the ontology prevents the conservative nominalist from doing.

Perhaps the conservative nominalist could invoke a fiction operator at this point (Field 1989). As also noted above, to gain verbal agreement with the Platonist, the conservative nominalist introduces a fiction operator of the form “According to [a suitable mathematical theory M] ... .” For example, in the case of the existence of infinitely many prime numbers, the conservative nominalist would state: “According to arithmetic, there are infinitely many prime numbers,” which is true and does not presuppose the existence of such numbers. In this way, the conservative nominalist is in a position to assert true mathematical statements without thereby being committed to the existence of the corresponding objects.

The trouble is that it now becomes clear that the truth of mathematical statements is relative after all, namely, to the fictional operator under use. It then follows for the conservative nominalist that mathematical statements, if true, are relatively true only—the very statement of mathematical relativism.

A different version of nominalism is offered by modal structuralism (Hellman 1989). According to the modal structuralist, each mathematical statement should be translated into a modal second-order language. Suppose that S states that there are infinitely many prime numbers. S should then be translated into two modal statements. According to the hypothetical component, if there were arithmetical structures of the appropriate kind, S would be true in such structures. According to the categorical component, the structures of that kind are possible. By translating mathematical statements in accordance with this schema, it is possible to preserve the truth-value of such statements without the commitment to mathematical objects. After all, only the possibility of certain structures is ever asserted. And by applying the translation schema to each step in a valid proof, the validity of the proof is preserved on the modal-structural interpretation. In this way, the truth of the theorems proved by Platonist means is also preserved—without the corresponding ontological commitments.

As a result, according to the modal-structural approach, each mathematical statement, if true, is relatively true only, that is, true relative to the modal-structural translation schema. Given that for the modal-structural nominalist, mathematical objects do not exist, if mathematical statements were taken literally, any such existentially quantified statement would be false, since the objects that are referred to do not exist. However, relative to the modal-structural interpretation, such mathematical statements, if true, would be true relative to the proposed translation schema. Thus, on this view, true mathematical statements are relatively true only—relative to the appropriate
translations. Once again, a form of mathematical relativism emerges from this version of nominalism.

Why do we get this outcome from the two formulations of nominalism just discussed? In a nutshell, the reason is because neither is able to take mathematical discourse literally. Given the need to reconstruct that discourse to avoid commitment to mathematical objects, fictional or modal operators are introduced. The upshot is that true mathematical statements are true only relative to suitable reformulations of the theories in question, and we obtain a form of mathematical relativism.

What would happen if the nominalist did not require any such reformulations of mathematical discourse? Would mathematical relativism be avoided then? Once again, it depends. I will consider two versions of nominalism that (seem to) take mathematical discourse literally, and argue that one has absolutist implications, whereas the other does not. In fact, I will examine a version of nominalism and a version of fictionalism (the distinction will become clear shortly). Absolutism, in this case, emerges from the nominalist front.

Consider deflationary nominalism (Azzouni 2004). According to the deflationary nominalist, mathematical objects do not exist. Only objects that are ontologically independent from our linguistic practices and psychological processes exist. This includes tables, chairs, mountains, human beings, genes, protons, and electrons. The criterion excludes, however, sets, functions, numbers, Mickey Mouse, and Sherlock Holmes. Mathematical objects, similarly to fictional entities, are entirely made up by us; they are the product of intentional acts of their authors. Hence, given the ontological independence criterion, these objects do not exist. Even though we quantify over mathematical objects, and cannot even formulate our best theories of the world without referring to these entities, such quantification does not require the existence of the latter, pace Quine (1960). The deflationary nominalist thus distinguishes quantification (even over entities that cannot be dispensed with) from ontological commitment. We regularly quantify over objects in whose existence we have no reason to believe, such as various kinds of fictional entities: talking mice, famous, but non-existent, detectives, or average moms who have 2.4 kids. Such quantification clearly does not require any commitment to the existence of the objects in question.

Deflationary nominalism does not entail mathematical relativism. According to the deflationary nominalist, mathematical statements are true (in a deflationary sense), but this does not require the existence of the corresponding objects, given that only ontologically independent entities exist – and this is not the case with mathematical objects. Furthermore, the underlying logic of mathematics is taken to be classical logic. So, the mathematical objects described are taken to be consistent and complete, in the sense that they do not have inconsistent properties, and for each property, either the objects in question have that property or lack it. On this view, mathematical statements are taken to be true simpliciter – not true relative to some interpretation or other, as the previous forms of nominalism required. What the deflationary nominalist relinquishes is the requirement that truth needs to have corresponding truth-makers. This piece of metaphysics is abandoned. Curiously, by rejecting this piece of metaphysics, deflationary nominalism obtains absolutist conclusions very similar to Fregean and Gödelian Platonism – minus the commitment to the existence of abstract entities. Mathematical truths are not relatively true on this view.
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Contrast deflationary nominalism to (agnostic) mathematical fictionalism (Buño 2008; 2009). According to the mathematical fictionalist, the issue of the existence of mathematical objects is simply left open. It is unclear that there is a non-question-begging way of settling the issue. And, interestingly enough, the issue need not be settled in order for us to make sense of mathematical practice. The quantifiers are read as not carrying ontological commitment. This much mathematical fictionalism shares with deflationary nominalism. But instead of adopting ontological independence as the mark of the real, as the deflationary nominalist does, the mathematical fictionalist simply suspends the judgment about the existence of mathematical objects. So the key difference between mathematical fictionalism and deflationary nominalism is that the latter denies the existence of mathematical objects, whereas the former is agnostic about the issue.

What does motivate such agnosticism? On the one hand, ontological independence from our linguistic practices and psychological processes is a property that Platonists have insisted that mathematical objects have. Numbers, functions, and sets would exist, according to the Platonist, even if no human being had ever thought of them. If the ontological independence criterion is adopted, Platonism seems to follow. On the other hand, what grounds do we have to believe that mathematical objects are indeed ontologically independent from us? Consider the mathematical principles that are introduced to characterize a given domain of objects. For instance, in arithmetic, we have that 0 is a natural number; the successor of a natural number is also a natural number, and these are all the natural numbers. As noted above, principles of arithmetic introduce the relevant concepts, and given a logic— which is typically classical, but need not always be— it is possible to operate with such concepts. Results are then proved based on these principles and the logical inferences that are invoked. The question then arises as to whether the concepts that are introduced by such principles correspond to independently existing objects—that is, objects that exist independently of any description offered by the relevant mathematical principles. Platonists will insist that they do. Nominalists insist that they do not. But it is not clear that there is a fact of the matter as to whether any of these views is ultimately correct. This motivates suspension of judgment about the issue, and an agnostic attitude about it. This is the path recommended by the agnostic mathematical fictionalist.

As a result, even though mathematical discourse is taken literally by the mathematical fictionalist, mathematical relativity does follow. After all, mathematical statements, if true, are relatively true only—relative to the relevant comprehension principles and the underlying logic in question. I think this is as it should be. How else could we make sense of the rich conceptual, structural, and logical plurality offered by mathematics?

6. Conclusion: The Significance of Mathematical Relativism

Mathematical relativism offers a pluralist account of mathematics that is able to make perfectly good sense of the plurality of different mathematical concepts, theories, frameworks, and logics that inform and are used throughout: mathematical practice. The proposal also challenges some of the traditional features that are commonly associated
with mathematics, such as its apparent necessity, its absoluteness, and its non-dynamic character.

On the mathematical relativist view, mathematical practice becomes a far more dynamic, diverse, and multifarious activity, with different theories, concepts, logics, and frameworks being simultaneously explored and articulated. Mathematics also becomes a more interesting, open-ended enterprise, without, however, losing its objectivity. After all, once a certain perspective is fixed — once certain mathematical principles are introduced and a given logic is accepted — it is no longer up to us whether certain results hold or not. It is then a matter of establishing the results (or their negation, or their independence) in the relevant framework. Given its non-dogmatic, pluralist character, mathematical relativism has much to offer.

References


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Further reading