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ON THE REFERENTIAL INDETERMINACY OF LOGICAL AND MATHEMATICAL CONCEPTS

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ABSTRACT. Hartry Field has recently examined the question whether our logical and mathematical concepts are referentially indeterminate. In his view, (1) certain logical notions, such as second-order quantification, are indeterminate, but (2) important mathematical notions, such as the notion of finiteness, are not (they are determinate). In this paper, I assess Field’s analysis, and argue that claims (1) and (2) turn out to be inconsistent. After all, given that the notion of finiteness can only be adequately characterized in pure second-order logic, if Field is right in claiming that second-order quantification is indeterminate (see (1)), it follows that finiteness is also indeterminate (contrary to (2)). After arguing that Field is committed to these claims, I provide a diagnosis of why this inconsistency emerged, and I suggest an alternative, consistent picture of the relationship between logical and mathematical indeterminacy.

KEY WORDS: finiteness, Henkin semantics, indeterminacy, second-order logic

1. INTRODUCTION

Are our logical and mathematical notions referentially indeterminate? That is, is the reference of logical and mathematical terms, or the extension of logical and mathematical predicates, indeterminate? The immediate response to this question is probably that none are. Although one would recognize the existence of referentially indeterminate terms in natural language, mathematical and logical vocabulary is typically taken as not having such terms. If anything is determinate, logical and mathematical notions are! (Let us call this response the “traditional view”.)

In some recent papers, Hartry Field has critically examined this response (Field, 1994, 1998). In his view, (1) certain logical notions, such as second-order quantification, are referentially indeterminate, but (2) important mathematical notions, such as the notion of finiteness, are not—they are indeed determinate. In this paper, I assess Field’s analysis, and argue that claims (1) and (2) turn out to be inconsistent. I then provide an alternative picture of the relationship between logical and mathematical indeterminacy.
Before proceeding, let me note that my criticism of Field’s account should not be taken as a defense of the traditional view. I think there is logical and mathematical indeterminacy, but not in the way advocated by Field.

2. FIELD’S ARGUMENT FOR THE INDETERMINACY OF SECOND-ORDER QUANTIFICATION

First, let’s rehearse Field’s argument for the indeterminacy of second-order quantification. His argument is basically an extension of Putnam’s well-known model-theoretic argument, which is meant to show that an ideal theory (a theory that is consistent, empirically adequate, simple, explanatory, etc.) cannot be false (see Putnam, 1976, 1980). According to Putnam, this shows that the metaphysical realist — for whom even an ideal theory can be false — is actually incoherent.

For our present purposes, we don’t need to consider the particular details of Putnam’s argument (see Douven, 1999 for a thoughtful discussion). The crucial point is that the argument relies on the (upward) Löwenheim–Skolem theorem, according to which if a first-order theory has an infinite model (a model with an infinite domain), then for every infinite cardinality, the theory has a model of that cardinality. As a result, there will always be nonstandard interpretations of that theory. These interpretations are nonstandard in the sense that they satisfy all the axioms of the theory in question (they are models of the theory, after all), but the extensions they assign to the predicates of the theory are not the usual ones. Consider, for example, first-order arithmetic. As is well known, given the Löwenheim–Skolem theorem, there are nonstandard models of this theory in which nonstandard ‘natural numbers’ that have infinitely many predecessors satisfy the predicate ‘natural number’. And this is the case despite the fact that ‘natural number’ is characterized by a condition entailing that a natural number only has finitely many predecessors. As a result, quantification over natural numbers becomes referentially indeterminate: both standard and nonstandard numbers satisfy the conditions that characterize ‘natural number’. First-order arithmetic cannot distinguish them, and in this sense, first-order quantification becomes indeterminate.

An obvious response to this argument is to complain that we didn’t have the right logic, and that is why we were led to this unfortunate conclusion. If we move to second-order logic, the Löwenheim–Skolem theorem fails, and so there are no nonstandard models to challenge the referential determinacy of our first-order quantifiers.
It is at this point that Field’s argument enters. What Field does is to extend Putnam’s indeterminacy argument to second-order theories. In other words, according to Field, if we suppose that Putnam is right about the referential indeterminacy of first-order theories with infinite domains, we need to grant that he will also be right about the referential indeterminacy of second-order quantification (see Field, 1994, pp. 394–396). That is, just by moving to second-order logic, we would not avoid the conclusion of Putnam’s indeterminacy argument. And so, if we think that first-order quantification is indeterminate (given Putnam’s argument), so is second-order quantification. But why is this the case?

Field’s argument – following Putnam (1980) – is that second-order logic has two different semantics. On the one hand, there is standard semantics, in which second-order variables range over all subclasses of the first-order domain of interpretation (and which doesn’t validate the Löwenheim–Skolem theorem). On the other hand, there is Henkin semantics, in which the range of second-order variables is a collection of not necessarily all the subclasses of the first-order domain. With Henkin semantics, the Löwenheim–Skolem theorem does hold for second-order logic (for details, see Shapiro, 1991). So, if Putnam’s argument establishes the referential indeterminacy of first-order quantification (in the case of natural numbers, for example), it should also establish the same result for second-order logic. After all, the formalism of this logic doesn’t uniquely determine the appropriate semantics for second-order logic. And so the second-order quantifier is indeterminate (between standard and Henkin semantics), in just the same way as the first-order quantifier is indeterminate (in the case of arithmetic) between standard and nonstandard (arithmetical) models. As Field points out:

What in our practice determines that our truth conditions are governed by the truth rule

(1) ($\forall X) A(X)$ is true in interpretation $J$ if and only if every subcollection of $J$ satisfies $A(X)$?

Why not

($1^*$) ($\forall X) A(X)$ is true in interpretation $J$ if and only if every $F$ subcollection of $J$ satisfies $A(X)$,

where $F$ is some property closed under second-order definability? Of course, if we were to state a truth rule for the second-order quantifier we would state (1) and not ($1^*$); but in virtue of what would our phrase ‘every subcollection’ mean every subcollection, rather than every $F$ subcollection? If the Putnam argument works in the first-order case, it works equally in this case; and its conclusion is of course not that we might mean ($1^*$) rather than (1), but that our second-order quantifier is indeterminate in its truth conditions […] Consequently, we do not really have a determinate conception of [standard] interpretation. (Field, 1994, pp. 395–396)
In other words, according to Field, if we accept Putnam’s model-theoretic argument for first-order theories, it is incoherent to reject the extension of the argument to second-order theories as well.

Of course, one could reject Putnam’s argument in the first place. In this case, second-order quantification may or may not be indeterminate. However, Field *endorses* Putnam’s argument, especially in the case of some first-order theories, such as set theory. He even *defends* the argument against a serious objection (Field, 1998, pp. 104–108). According to this objection, it would be *incoherent* to suppose that our notions of set and second-order quantification are indeterminate. After all, the objection goes, if we admit a disquotational account of truth, ‘s is a set’ has perfectly determinate truth conditions: ‘s is a set’ is true if and only if s is a set. (Similarly for second-order quantification.) So, it would be incoherent to suppose that sentences referring to sets and employing second-order quantification are indeterminate.

Field responds to this objection by arguing that referential indeterminacy and vagueness have a lot in common, and by noting that we can make sense of referential indeterminacy in the same way as we can make sense of vagueness. After all, both in the case of vagueness and in the case of indeterminacy, the problem is not to know whether the sentence ‘s is a set’ is true, but rather whether ‘s is a set’ is *definitely* true. For one to overcome the indeterminacy, what is required is the existence of *definite* truth-conditions for something to be a set. And, of course, the disquotational schema *doesn’t* entail that *s* is *definitely* a set, or that *s* is *definitely* not a set. So, if we cannot assert that the sentence ‘s is a set’ has *definite* truth-conditions, this indicates that the notion of set is referentially indeterminate.³ (Again, a similar point applies to second-order quantification.) So, according to Field, we can *make sense* of the notion of referential indeterminacy, just as we can make sense of vagueness. There is *no incoherence* in the claim that our notions of set and second-order quantification are indeterminate.

Given Field’s endorsement of Putnam’s argument, the conclusion follows that, in his view, second-order quantification is indeed indeterminate.

3. FIELD’S ARGUMENT FOR THE DETERMINACY OF THE NOTION OF FINITENESS

The second step in Field’s argument is to claim that there is a substantial *difference* between second-order quantification and the notion of finite-
ness. Even if second-order quantification is indeterminate (given Putnam’s model-theoretic argument), the notion of finiteness is \textit{not} indeterminate. As Field points out, the claim that we lack a determinate conception of finiteness is really difficult to accept:

This conclusion [that the notion of finiteness is not determinate] seems prima facie much harder to swallow than the analogous conclusion about the notion of set or second-order quantification. It seems harder because the notion of finiteness (or natural number) seems much more elementary, and because it seems presupposed even by our notion of inferential procedure. After all, an inference, or proof from given assumptions, is a \textit{finite} sequence of expressions in which each is either a premise or a logical axiom or follows from predecessors in the sequence by certain rules; \textit{if we have no determine notion of finite, then there is going to be an indeterminacy even in our notion of proof from given assumptions}. And thus it seems we can’t even talk coherently about what our inferential procedure that supposedly fixes our indeterminate truth-conditions is. Surely, it would seem, we have talked ourselves into an unacceptable situation. (Field, 1998, p. 104; italics added)

But why would anyone claim that the notion of finiteness is indeterminate? The reason derives, once again, from Putnam’s model-theoretic argument (Field, 1998, p. 103). Suppose that we characterize the notion of finiteness by defining the quantifier ‘there are only finitely many’ in set-theoretic terms:

\[ \exists_{\text{fin}} x A(x) \text{ iff } \{ x : A(x) \} \text{ is in 1-1 correspondence with the predecessors of some natural number } n. \]

It then follows, by Putnam’s argument, that there will always be nonstandard interpretations of arithmetic for which many formulas of the form \( \exists_{\text{fin}} x B(x, y) \) will be satisfied by objects \( c \) for which there are \textit{infinitely many} \( b \) such that \( (b, c) \) satisfies \( B(x, y) \). In particular, Field argues, ‘natural number’, though defined by a condition entailing ‘Each natural number has only finitely many predecessors’, will be satisfied in some interpretations of our overall theory of numbers by nonstandard ‘natural numbers’ with infinitely many predecessors; and ‘finite sequence of expressions’ (sequence with only finitely many initial segments) will be satisfied in some interpretation of our overall theory of finite sequences by nonstandard ‘finite sequences’ with infinitely many initial segments. (Field, 1998, p. 103)

He then concludes:

And so if the basic Putnamian argument is correct, it looks as if we can’t have a determinate conception of finiteness, or a conception of the natural numbers that is determinate even up to isomorphism. (Field, 1998, pp. 103–104)

However, as noted above, Field takes this conclusion to be unacceptable. It will make our inferential procedures completely indeterminate (given that such procedures ultimately depend on the notion of finiteness). In order to resist this drastic conclusion, Field devises two strategies to
guarantee the determinacy of the notion of finiteness. I shall consider them in turn.

The First Strategy: Blocking the use of Putnam’s Argument. Field re-examines Putnam’s argument, and concludes that ultimately it cannot be used to show that the notion of finiteness is indeterminate. This is because Putnam’s argument depends on the downward version of the Löwenheim–Skolem theorem, and a downward Löwenheim–Skolem argument won’t give you a nonstandard interpretation of ‘finite’: finite sets satisfy ‘finite’ in every interpretation in which the axioms of set theory come out true, since 1-1 correspondences between finite sets are themselves finite and hence exist in every such interpretation. (Field, 1998, pp. 122–123)

And without a nonstandard interpretation of ‘finite’, there is no referential indeterminacy of the sort envisaged by Putnam. In this case, the use of Putnam’s argument is blocked.

However, instead of using the (downward) Löwenheim–Skolem theorem to obtain a nonstandard interpretation, one could use a compactness argument. This is certainly a way of generating nonstandard models. The problem, according to Field, is that the models that are generated by a compactness argument are highly “unintended”, in that even the physical vocabulary of the theory in question will be open to reinterpretation. So, for example, expressions that relate physical events, such as ‘earlier than’ or ‘at least one second apart’, will be satisfied in some interpretations by pairs of events such that the first is not earlier than the second, or which are not one second apart. In other words, such interpretations do not provide “the slightest reason to think that the constraints on the physical vocabulary will be preserved” (Field, 1994, p. 416; see also Field, 1998, p. 122). And given that such constraints are not preserved, we have reason to reject the resulting nonstandard models as blatantly inadequate. This indicates that without the (downward) Löwenheim–Skolem theorem, Putnam’s result cannot be used to establish that the notion of finiteness is indeterminate after all.

The Second Strategy: Using Brute Force. Even if Putnam’s argument doesn’t show the indeterminacy of ‘finiteness’, it is still possible that the standards imposed by Putnam to determine the extension of ‘set’ and ‘finite’ are enough to prevent ‘finite’ and ‘set’ from having a determinate extension (Field, 1994, p. 416; see also Field, 1998, p. 123). In order to claim that this is not the case, Field indicates how the extension of ‘finite’ can actually be determined. His idea is to determine the extension of ‘finite’ by appealing to the physical world.

In order to do that, Field defines a one-place predicate $B(Z)$ as follows (ibid.):
(A) Suppose that Z is a set of events such that (i) Z has an earliest member and a latest member, and (ii) any two of Z’s members occur at least one second apart.

(B) Moreover, suppose that there is no finite bound on the size of the sets that satisfy Z.

Field calls the above conditions his “cosmological assumptions”. The idea is that if such conditions are satisfied by events in the physical world, then the notion of finiteness can be determined. After all, from (A) and (B), and some set theory, it follows that:

\[(*) \forall Y (Y \text{ is a set } \supset (Y \text{ is finite } \equiv \exists Z \exists f (B(Z) \& f \text{ is a function that maps } Y \text{ 1-1 into } Z))).\]

With (*), it becomes clear that the extension of the notion of finiteness can be determined: one adopts the set Z of physical events as a “physical standard” of finiteness, as it were. Field’s conclusion is that, as opposed to second-order quantification, the notion of finiteness is indeed determinate.

4. THE PROBLEM WITH FIELD’S ACCOUNT

Let us take stock. As we saw, according to Field:

(1) certain logical notions, such as second-order quantification, are indeterminate,

but

(2) some mathematical notions, such as the notion of finiteness, are determinate.

The problem with this account is that it turns out to be inconsistent. Let us see why.

First, note that the notion of finiteness can only be properly characterized – I emphasize, properly characterized – in pure second-order logic; finiteness is ultimately a second-order notion (see Shapiro, 1991, pp. 100–106). The reason for this claim is that (a) the notion of infinity cannot be properly characterized in first-order logic. Given the existence of non-standard models of first-order theories, the characterization of infinity in first-order logic allows that ‘finite’ be satisfied by infinite sets. So, Field must accept that one cannot properly characterize infinity in first-order logic. However, (b) in pure second-order logic, the notion of infinity can be properly characterized, and it has no such unintended consequences. Given that typically the models of a second-order theory are isomorphic,
‘finite’ is only satisfied by finite sets. As a result, the notion of finiteness is properly second-order.

It might be argued that we need not go to full second-order logic to characterize the notion of finiteness; logics that are considerably weaker than full second-order logic can be used to characterize this notion. Strictly speaking this is, of course, correct. But unless we have a characterization of ‘finiteness’ that doesn’t allow for non-standard models, we won’t have properly characterized that notion. After all, if the logic in question doesn’t exclude non-standard models, it may still be possible that sets with infinitely many members satisfy the predicate ‘finite’, and thus the proposed characterization won’t be adequate. (I’ll elaborate on this point below.) Clearly first-order logic doesn’t exclude such non-standard models, and so it fails to provide an adequate characterization of ‘finite’. Of course, there are logics that will do the trick which are not as strong as full second-order logic (such as dyadic second-order logic). But to be adequate for the task at hand, these logics will need to have categorical models, and they end up being close enough to second-order logic that they deserve to be called ‘second-order’.

Two important consequences follow from the above discussion:

(i) if Field is right in claiming that second-order quantification is indeterminate (see (1), above), given that finiteness is only adequately characterized in pure second-order logic, it follows that finiteness is also indeterminate (contrary to (2)).

This is because the indeterminacy of second-order quantification – the indeterminacy between standard and Henkin semantics – ultimately applies to the characterization of the notion of finiteness. But why is this so?

To characterize the notion of finiteness in second-order logic, two kinds of models of the basic mathematical theory that we use need to be considered: standard (second-order) models (models from standard semantics, which will be denoted by ‘$M^*$‘), and Henkin models (models from Henkin semantics, which will be denoted by ‘$M$‘). We may initially think that each of these models yields exactly the same extension for ‘finite’, and so finiteness turns out to be a determinate notion – even though it’s defined in terms of a notion (second-order quantification) that is not determinate. Here is an argument that may initially persuade us of that:

Suppose that there are exactly $n$ Fs in $M^*$. Then for some number $n$, there is a sentence of the first-order predicate calculus with identity according to which ‘There are exactly $n$ Fs’ and this sentence is true in $M^*$. Moreover, it is also a theorem that if there are exactly $n$ Fs, then there are finitely many $F$s. Thus, ‘There are finitely many $F$s’ is true in $M^*$. Conversely, if ‘There are finitely many $F$s’ is true in $M^*$, then (the
formalization of) 'For some natural number $n$, there is a bijection between the $F$s and the numbers less than $n$' is true in $M^*_s$. After all, the following biconditional is also a theorem: 'There are finitely many $F$s if for some natural number $n$, there is a bijection between the $F$s and the numbers less than $n$'. Now in moving from the standard model $M^*$ to the Henkin model $M$, we only remove some functions and relations from the domain of the second-order variables. No new functions or relations are added to the domain. So, if according to $M^*_s$, $g$ is a bijection between the $F$s and the natural number $n$, $g$ plays the same role in $M$. Hence the collection of $F$s is really finite. As a result, it might be thought that 'finite' is determined, despite the indeterminacy of second-order quantification.

The problem with the above argument is that when we use Henkin semantics, given that the (upward) Löwenheim–Skolem theorem holds, there will be nonstandard models of our basic mathematical theory in which sets with infinitely many members will satisfy 'finite' (the argument for this claim will be discussed in the next section). But this is not the case in standard semantics, where the Löwenheim–Skolem argument fails. As a result, despite appearances, the notion of finiteness is not determined in the same way by each semantics. Thus, if second-order quantification is indeterminate (between standard and Henkin semantics), this indeterminacy also applies to the characterization of finiteness.

It might be argued that the second-order quantifier can be partially determinate, allowing non-standard models, but only non-standard models in which finitude behaves in the standard way. The trouble here is to specify the conditions that guarantee that "finite" behaves standardly in all of the non-standard models. For this to succeed as a response to Putnam's argument, this characterization needs to be done without assuming that the notion of finiteness has already been determined. And it's not clear how this could be done. In fact, as I argue in the next section, this is precisely the problem with Field's "brute force" move.°

The second consequence that follows from the above discussion is this:

(ii) if contrary to (i) second-order quantification is determinate, given that all we need to formulate the notion of finiteness is second-order logic, it follows that the notion of finiteness also becomes determinate.

Of course, Field wants to endorse the consequent of (ii) (see (2)), but he cannot accept the antecedent (given that he accepts (1)). As a result, Field would have to accept (ii). Indeed, as he points out:

it seems to me that there is no serious possibility of holding [...], second-order quantification determinate and finiteness not. (Field, 1998, p. 104, note 8)

Combining (i) and (ii), we have the following biconditional:
(3) Second-order quantification is indeterminate if and only if the notion of finiteness is also indeterminate.

Given that Field asserts the left-hand side of the above biconditional (see (1)) and denies the right-hand side (see (2)), his analysis is inconsistent. For it follows from (1)–(3) that finiteness is indeterminate and is not indeterminate.

5. An Alternative Relationship between Logical and Mathematical Indeterminacy

My diagnosis of the situation is that what led Field to stumble into this inconsistency are his arguments for the determinacy of the notion of finiteness. But as I will try to indicate now, none of his arguments provide compelling reasons for the determinacy of the notion of finiteness. So, Field has no reason to assert (2), above.

The Problem with the Attempt to Block the Use of Putnam’s Argument. As we saw, the crucial move of Field’s first strategy to resist Putnam’s argument was to claim that the downward Löwenheim–Skolem theorem doesn’t yield a nonstandard interpretation of ‘finite’. After all, finite sets satisfy ‘finite’ in every interpretation of set theory in which its axioms are true.

The problem with this suggestion is that it fails to block Putnam’s argument. Even granting that finite sets do satisfy ‘finite’ in every interpretation of set theory, and that the downward Löwenheim–Skolem theorem fails to generate nonstandard interpretations of ‘finite’, Putnam’s argument still stands. To block the argument, what needs to be established is that (in any interpretation) only sets that have finitely many members satisfy ‘finite’. But given the upward Löwenheim–Skolem theorem (which is based on the compactness theorem), there will always be nonstandard interpretations in which ‘finite’ is satisfied by sets with infinitely many members. After all, we can use the upward Löwenheim–Skolem theorem (or the compactness result) to produce elementary extensions of the (standard) model of the theory in question with nonstandard integers, and use the latter integers to play new roles in the model (for example, to be in the extension of ‘finite’). As a result, in the model, sets with infinitely many members will satisfy ‘finite’.

Field’s second argument to block Putnam’s result was to consider another strategy to generate nonstandard models, namely, via the compactness result. As we saw, the compactness result also yields nonstandard
interpretations that support Putnam’s argument. Field acknowledges the
point, but he complained that the nonstandard interpretations generated by
the compactness theorem violate the constraints on the physical vocabulary
(for example, in some interpretations, the extension of ‘event’ will contain
things that are not events). And indeed some nonstandard interpretations
will violate these constraints. But this fact needs to be used carefully.
As we will see in the next section, in pressing this point, Field begs the
question.

The Problem with the Use of Brute Force. Let us recall the crucial feature
of Field’s second strategy to claim that the notion of finiteness is determi-
nate. He introduced two “cosmological assumptions”, and in terms of these
assumptions, the determinacy of ‘finiteness’ was to be established. As we
saw, the “cosmological assumptions” were:

(A) Suppose that \( Z \) is a set of events such that (i) \( Z \) has an earliest member
and a latest member, and (ii) any two of \( Z \)’s members occur at least
one second apart.

(B) Moreover, suppose that there is no finite bound on the size of the sets
that satisfy \( Z \).

The problem with this move is that the notion of finiteness occurs in the
assumption (B). As a result, we can easily apply Putnam’s argument to set
theory (which is presupposed in Field’s discussion) plus Field’s “cosmo-
logical assumptions”, and generate a nonstandard interpretation in which
‘finite’ is satisfied by nonstandard sets that are infinite. So, Field’s assump-
tions fail to determine the extension of ‘finite’, and the notion of finiteness
becomes referentially indeterminate.

Field acknowledges this point. However, in his view, there is something
deeply mistaken about the nonstandard models that satisfy his cosmologi-
cal assumptions:

If our first cosmological assumption is true [see (A), above], any […] model [in which
certain infinite sets satisfy the predicate ‘finite’] must assign a nonstandard extension
to the formula \( B(Z) \); and in particular, it must either contain things that satisfy ‘event’ which
are not events, or it must contain pairs of events which satisfy ‘earlier than’ or ‘at least
one second apart’ even though the first is not earlier than the second or the two are not
one second apart. Either way, the model will violate the constraint on the interpretation
of the physical vocabulary, viz. that the extension of such a predicate in the model can only
contain things that actually have the corresponding property. (Field, 1994, p. 417; italics
omitted)

This leads Field to conclude that

no model of \( S \) [that is, of set theory plus Field’s “cosmological assumptions”] in which
‘event’ and ‘earlier than’ and ‘at least one second apart’ satisfy the constraints on the
interpretation of the physical vocabulary can be one where any infinite sets satisfy $B(Z)$. 
(Field, 1994, p. 417; italics omitted)

And so

no allowable model of $S$ can be one where any infinite sets satisfy ‘finite’. (Field, 1994, 
p. 417; italics omitted)

But what are the “allowable models of $S$”? That is, what are the allowable models of set theory plus the “cosmological assumptions”? From the quotations given above, it is clear that by ‘allowable model’ Field means a model that satisfies the constraint on the physical vocabulary. So, an allowable model is a model in which the extension of a predicate in the model only contains things that actually have the corresponding property. Thus, if in a given model the extension of the predicate ‘event’ contains things that are not events, this model is not “allowable”. This is, of course, a quite plausible requirement.

The problem is that to use the notion of ‘allowable model’ to exclude nonstandard interpretations in the way Field wants begs the question. After all, Field’s second cosmological assumption, (B), requires that there be no finite bound on the size of the sets that satisfy $Z$. So, in order to claim that a given model of $S$ is not “allowable”, because in this model ‘finite’ is satisfied by certain infinite sets, one needs to assume that the extension of ‘finite’ is already determined. To see why this is the case, note that according to Field, we shouldn't have any difficulty to determine whether the extension of a physical predicate in a given model only contains things that actually have the corresponding property. We just “look” at the extension of the physical predicate – say, ‘event’ – to determine whether the things in that extension are indeed events. Of course, this was the reason why Field introduced a physical standard of finiteness to articulate his cosmological assumptions. However, given that ‘finite’ occurs in the second “cosmological assumption”, to guarantee that we only have allowable models of $S$, it is required that the extension of ‘finite’ in our models only contains finite things. But this would only be the case if the extension of ‘finite’ had already been determined – which is exactly the point in question!

Of course, Field’s idea is to determine the extension of ‘finite’ as the result of determining the extension of the predicates in the “cosmological assumptions” (and these predicates have their extension determined via their connection with the physical world). The idea is ingenious. But it ultimately fails, since ‘finite’ occurs in the second “cosmological assumption”, and so this predicate needs to be interpreted in any model of $S$. And to exclude nonstandard interpretations of ‘finite’ because they are not “allowable” presupposes that the extension of ‘finite’ has already been
determined. Thus, the extension of ‘finite’ cannot be determined by Field’s procedure without begging the question.

Note that I am not claiming that Field intended to show that there is a physically definable omega-sequence satisfying his “cosmological assumptions”. Whether there is such a sequence or not is a cosmological fact which surely cannot be proved on a priori grounds, and which would be extremely difficult to prove experimentally. Field’s idea is that if the physical world cooperates in the right ways, an omega-sequence will be physically definable. My point is that even if the physical world cooperates and there is such an omega-sequence, the latter won’t do the required job that Field expects it to do, for the reasons given in the paragraphs above.

It might also be complained that Field’s argument only requires that the “cosmological assumptions” obtain, it doesn’t require that we have a way of formulating these assumptions that is resistant to Putnam’s argument. The problem with this complaint is not hard to see. Suppose we try to block Putnam’s paradox by advancing a particular strategy $Y$ (in Field’s case, by formulating some “cosmological assumptions”). If Putnam’s argument can be applied to $Y$, then $Y$ fails to block the argument. As argued above, if Putnam’s argument can be applied to the cosmological assumptions, then the latter fail to determine the extension of “finite”, and Putnam’s original problem simply comes back again.

Logical and Mathematical Indeterminacy Related. Given the inconsistency of Field’s analysis, how should we conceive of the relationship between logical and mathematical indeterminacy? The argument used to establish the inconsistency of Field’s account already indicates one way of understanding that relationship, a way that emphasizes the close connection between these notions: logical and mathematical indeterminacy stand or fall together.

This is one aspect of the traditional view (about logical and mathematical indeterminacy) that we can preserve safely. One would expect a close connection between the determinacy (or indeterminacy) of mathematical and logical notions. So, if those notions turn out to be indeterminate, one would expect both types of notion to be indeterminate. Thus, in order to remain consistent, the best strategy is to admit the close link between mathematical and logical indeterminacy. This link is clearly expressed by claim (3) discussed above, namely:

(3) Second-order quantification is indeterminate if and only if the notion of finiteness is also indeterminate.

It is now clear that there is something right about the traditional view after all: the determinacy (or indeterminacy) of logical and mathematical
notions is closely connected. But there is also something deeply wrong with this view, and Field’s analysis made this point forcefully: our logical and mathematical notions may be indeterminate after all. But if this is the case, both such notions (second-order quantification and finiteness) will be indeterminate, not only one of them.

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NOTES

1 Putnam’s argument also relies on the downward Löwenheim–Skolem theorem, according to which if a first-order theory has an infinite model, then it has a denumerable model (a model whose domain is denumerable).

2 Field then notes that Weston made this point in his (1976).

3 Similarly, if ‘a is bald’ lacks definite truth-conditions, this indicates that ‘bald’ is a vague notion.

4 According to the compactness theorem, if \( \Gamma \) is a set of sentences, and if every finite subset of \( \Gamma \) has a model, then \( \Gamma \) has a model.

5 For example, the usual definition of (Dedekind) infinite — according to which there is a one-to-one function \( f \) from a set \( X \) to \( X \) whose range is a proper subset of \( X \) — can be characterized in pure second-order logic as follows:

\[
\exists f ( \forall x \forall y (f x = f y \rightarrow x = y) \& \forall x (X x \rightarrow X f x) \& \exists y (X y \& \forall x (X x \rightarrow f x \neq y))).
\]

This immediately yields a characterization of finiteness: \( X \) is finite iff \( X \) is not infinite in the above sense (for a discussion, see Shapiro, 1991, pp. 100–102).

6 In other words, second-order theories are categorical. This is the case, for example, of second-order arithmetic and analysis. In the case of second-order set theory, if two models are not already isomorphic, then one is isomorphic to an “initial segment” of the other (for details, see Shapiro, 1991, pp. 82–86).

7 Throughout this discussion, I am assuming with Field that a natural way to render the notion of finitude determinate is by characterizing it explicitly in terms of first- or second-order languages (and showing that first- or second-order quantification is determinate). This assumption could be challenged, of course. But given that Field seems to grant it, I will work with it. Note that no generality is lost with the assumption, given that attempts to characterize the notion of finiteness that do not explicitly rely on first- or second-order languages (say, by exploring natural language) can be translated into first- or second-order languages. Thus, the assumption is safe.
REFERENCES


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