Introduction

Philosophy of Mathematics: Old and New

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This volume contains essays by 13 up-and-coming researchers in the philosophy of mathematics, who have been invited to write on what they take to be the right philosophical account of mathematics, examining along the way where they think the philosophy of mathematics is and ought to be going. As one might expect, a rich and diverse picture emerges. Some broader tendencies can nevertheless be detected: there is increasing attention to the practice, language, and psychology of mathematics, a move to reassess the orthodoxy, as well as inspiration from philosophical logic.

In order to describe what is new in a field, we need some understanding of what is old. We will accordingly begin with a quick overview of earlier work in the philosophy of mathematics before attempting to characterize the tendencies that emerge in this volume and describing how the various essays fit into this pattern.

The beginning of modern philosophy of mathematics

Modern philosophy of mathematics can plausibly be taken to have begun with the pioneering work of Gottlob Frege in the late nineteenth century. Having first developed modern formal logic—essentially what we now know as higher-order logic—Frege was able to articulate and pursue his philosophical goals with unprecedented precision and rigor. His primary goal was to show that arithmetic and analysis are just "highly developed logic", that is, that all the basic notions of these branches of mathematics can be defined in purely logical terms and that, modulo these definitions, mathematical theorems are nothing but logical truths. This would show that large parts of mathematics are analytic rather than synthetic a priori, as Kant so influentially had argued. Although later philosophers of mathematics have often disagreed with Frege's substantive views, the precision and rigor which Frege introduced into the debate have remained a widely accepted ideal.

Another great influence on modern philosophy of mathematics came from the logical paradoxes that were discovered in the late nineteenth and
early twentieth century. An early casualty was the foundation for mathematics that Frege himself had proposed, which Russell’s paradox revealed to be inconsistent. More generally, the paradoxes prompted a broad examination of foundational and philosophical questions concerning mathematics to which both philosophers and mathematicians contributed.

This examination led to the articulation of what we now regard as the three classical philosophies of mathematics. Logicians agree with Frege that much or all of mathematics is reducible to pure logic. But because of the paradoxes they tie this claim to a carefully circumscribed system of logic. The classical development of this view is found in Russell and Whitehead’s famous *Principia Mathematica*. Intuitionists disagree with the logicians and argue that the only legitimate foundation for mathematics is provided by pure intuition and mental constructions. This philosophical conception of mathematics is given a sophisticated development by L.E.J. Brouwer and Arend Heyting, who show that it leads to a revision not just of classical mathematics but even of the classical logic developed by Frege. Finally, formalism was developed and defended by the great mathematician David Hilbert. On the standard interpretation of this view, only finitary mathematics has content—which it obtains from pure intuition—whereas infinitary mathematics is a formal game with uninterpreted symbols. Hilbert’s goal was to prove the consistency of infinitary mathematics on the basis of finitary mathematics alone. This goal is shown to be unattainable by Gödel’s second incompleteness theorem, which says roughly that any consistent formal system of a certain mathematical strength cannot prove its own consistency, let alone that of stronger systems.

The recent past

Until around 1960, the discussion of the three classical philosophies of mathematics and their descendants dominated much of the debate (although Wittgenstein’s views too received much attention). In the three decades that followed, much of the agenda was set by W.V. Quine and Paul Benacerraf.

According to Quine, mathematics isn’t essentially different from theoretical physics. Both go beyond what can be observed by means of our unaided senses. And both are justified by their contribution to the prediction and explanation of states of affairs that can be thus observed. Sets are therefore said to be epistemologically on a par with electrons. In this way Quine uses his holistic form of empiricism to support a platonistic interpretation of mathematics, according to which there really are abstract mathematical objects. This approach relies on what has become known as the “indispensability argument”, which was explicitly articulated and developed by Hilary Putnam. According to this argument, we have reason to believe in the existence of sets, just as we have reason to believe in the existence of electrons, because both sorts of objects are indispensable to our best scientific theories of the world.

Benacerraf’s influence is due primarily to two seminal articles. In “What Numbers Could Not Be” (1965), Benacerraf challenges all attempts to identify the natural numbers with sets and argues (following Dedekind and others) that all that matters to mathematics is the structure of the natural number system, not any “intrinsic properties” of these numbers. Considerations of this sort have given rise to various forms of *mathematical structuralism*. In particular, Geoffrey Hellman has developed an eliminative form of structuralism according to which mathematics is concerned with the structures that are and could be instantiated by concrete objects,¹ whereas Charles Parsons, Michael Resnik, and Stewart Shapiro defend non-eliminative forms of structuralism which take mathematical objects to be real yet to be nothing more than positions in structures.

Platonistic views have met with epistemological challenges for as long as they have been around. Benacerraf’s second paper, “Mathematical Truth” (1973), articulates a striking version of an epistemological challenge to platonism about mathematics. If mathematics is about abstract objects, how is mathematical knowledge possible? Knowledge of some class of objects typically requires some form of contact or interaction with these objects. But if mathematical objects are abstract, no such contact or interaction seems possible. Many philosophers (most famously Hartry Field) have regarded Benacerraf’s challenge as an argument for *nominalism* (that is, the view that there are no abstract objects). The quarter century following the publication of “Mathematical Truth” saw a great variety of nominalistic accounts of mathematics, associated with Field, Hellman, Charles Chihara, and others. Field’s nominalism was also meant to respond to the indispensability argument by showing how to reformulate scientific theories without quantification over mathematical objects.

The contemporary debate

The views defended by Quine and the challenges so powerfully formulated by Benacerraf are still with us. The indispensability argument is still being debated and has recently been defended by Mark Colyvan and sympathetically discussed by Alan Baker. Although mathematical structuralism has received more criticism than support over the past decade, it has no doubt secured a position as one of the canonical philosophies of mathematics. And nominalism continues to exert a strong influence on many philosophers.

The past 10–15 years have seen various innovations and new orientations as well. New forms of fictionalism and nominalism have been developed by Jody Azzouni, Stephen Yablo, and others, aimed at doing better justice to actual mathematical language and thought than traditional forms of error
theory or fictionalism. There has also been a revival of interest in logicism, where a neo-Fregean view has been defended by Bob Hale and Crispin Wright and received careful philosophical scrutiny and technical analysis by George Boole and Richard Heck.

A particularly important new orientation goes by the name of "naturalism". The starting point is Quinean naturalism, which rejects any form of "first philosophy" in favor of a conception of philosophy as informed by and continuous with the natural sciences. Naturalist approaches to mathematics share Quine's disdain for any form of philosophy which aspires to be prior to or more fundamental than some successful scientific practice. John Burgess, Gideon Rosen, and especially Penelope Maddy have argued that mathematical questions allow of no deeper form of justification than what is operative in actual mathematical practice. This has led to increased attention to the practice and methodology of mathematics, exemplified for instance by the work of Maddy and Paolo Mancosu.

All of the concerns described in this section are represented in the present volume. Another theme, which emerges in this volume is an increasing engagement with neighboring disciplines such as linguistics, psychology, and logic. In what follows, we will attempt to place the contributions to this volume in a larger intellectual landscape. Although the categories with which we operate are not sharply defined, and although many of the essays discuss questions belonging to several of these categories, it is our hope that this will provide readers with a useful overview. These categories have also been used to structure the collection into five corresponding parts.

Part I. Reassessing the orthodoxy in the philosophy of mathematics

The two chapters that make up Part I of the volume critically examine the orthodoxy in the philosophy of mathematics.

Roy Cook's essay explores and defends Frege's combination of logicism and mathematical platonism. He also argues that the neo-Fregean view defended by Hale and Wright shares more features of the original Fregean view than is generally realized; in particular, that both Frege and the neo-Fregeans articulate an epistemic notion of analyticity (unlike Kant's and Quine's semantic notions) and seek to show that much of mathematics is analytic in this sense. For Fregeans old and new, to be analytic is to be justifiable on the basis of logic and definitions alone and thus to have absolutely general validity. The chief difference is that, for Frege, "logic" includes not just second-order logic but also the notorious Basic Law V, whereas for the neo-Fregeans, "definitions" include not just explicit definitions but also implicit ones such as abstraction principles.

According to conventional wisdom, Benacerraf established that numbers cannot be sets. Alexander Paseau challenges this view, exploring and defending the opposing Quinean view that the numbers can be reduced to sets. This view enjoys ontological, ideological, and axiomatic economy. But it also faces various objections, which Paseau discusses and rejects. Perhaps the most serious objection is that a reductionist interpretation is incompatible with what speakers mean with their own arithmetical vocabulary. Paseau offers two responses. Firstly, he denies that speakers have transparent knowledge of the referents of their words, in arithmetic or elsewhere. Secondly, he argues that any "hermeneutic" concern with getting actual meanings right must be secondary to the overall theoretical virtues of a proposed account of arithmetic.

Part II. The question of realism in mathematics

The question of realism and anti-realism in mathematics has been with us for a long time and is unlikely to go away any time soon. The contributions to Part II reassess the question and its significance, while developing new approaches to the problems that are raised.

Otávio Bueno's contribution is a defense of mathematical fictionalism. He identifies five desiderata that an account of mathematics must meet in order to make sense of mathematical practice. After arguing that current versions of platonism and nominalism fail to satisfy the desiderata, he outlines two versions of mathematical fictionalism, which meet them. One version is based on an empiricist view of science and has the additional benefit of providing a unified account of both mathematics and science. The other version is based on the metaphysics of fiction and articulates what can be considered a truly fictionalist account of mathematics. Bueno argues that both versions of fictionalism satisfy the desiderata and he thinks that they are best developed together. He concludes that mathematical fictionalism is alive and well.

Peter Koellner discusses the question of pluralism in mathematics. A radical pluralist position is defended by Carnap, who regards it as merely a matter of practical convenience, rather than objective truth, whether we should accept certain quite elementary arithmetical statements (so-called $\Pi^0_1$-sentences). Koellner critiques Carnap's position and argues that his radical form of pluralism is untenable. To address the question whether a more reasonable pluralism might be defensible, he develops a new approach to the question of pluralism inspired by a comparison with physics. In both cases, we have a certain pool of "data": respectively observational generalizations and $\Pi^0_1$-sentences. Two theories are "observationally equivalent" if they have the same entailments concerning "the data". We then face the "problem of selection": which of the many incompatible theories that are "observationally adequate" should we select as true? Koellner describes some
constraints on the problem of selection which arise from set theory itself and argues that these constraints rule out the common view that the set-theoretic independence results alone suffice to secure pluralism. The chapter ends with a discussion of a “bifurcation scenario”, that is, a scenario about possible developments in set theory which would arguably support a form of set-theoretic pluralism. This scenario is based on a recent result with Hugh Woodin and has the virtue that whether it pans out is sensitive to actual developments in mathematics.

Mary Leng’s contribution is a critical discussion of “algebraic” approaches to mathematics. This class of approaches is explained in terms of the classic Frege–Hilbert debate, where Frege argued that mathematical axioms are assertions about some mathematical reality, and Hilbert had the more algebraic view that axioms rather define or circumscribe their subject matter. Some of the major philosophical approaches to mathematics belong to the algebraic camp. Leng argues that these algebraic approaches are able to address Benacerraf’s two challenges to mathematical platonism but that the algebraic approaches face two new challenges, namely explaining the modal notions on which they crucially rely, and accounting for mixed mathematical–empirical claims.

Part III. Mathematical practice and the methodology of mathematics

Making sense of mathematical practice and the methodology of mathematics are crucial components of a philosophical account of mathematics. Intriguing issues emerge when these components are taken seriously: from the role of mathematical explanation in mathematical practice, through the issue of the status of inconsistent mathematical theories, to the complexities of understanding applied mathematics. These issues are discussed in Part III.

Alan Baker’s chapter opens by calling attention to the increased interest among philosophers of mathematics in the concept of mathematical explanation, which has long been ignored. A central thesis of the chapter is that the notion of mathematical explanation can be illuminated by a study of the concept of an accidental mathematical fact. Since mathematical facts are assumed to be necessary, mathematical accidents cannot be explained in terms of the notion of contingency. Baker first explains the notion of “a mathematical coincidence” in terms of the equality of two quantities lacking a unified explanation. A mathematical truth is then said to be accidental to the extent that it lacks a unified, non-disjunctive proof.

Mark Colyvan discusses some problems posed by inconsistent mathematical theories. For instance, naive set theory is inconsistent, and arguably, so is eighteenth century calculus. How can reasoning in such theories be represented? Clearly, we don’t proceed to derive every sentence whatsoever, as is permitted in classical logic. Paraconsistent logics provide a way of avoiding this trivialization. Such logics enable Colyvan to ask some philosophical questions. Firstly, what happens when the Quinean indispensability argument is applied to inconsistent but nevertheless useful mathematical theories? Colyvan suggests that theorists may then have had reason to believe in the existence of inconsistent mathematical objects. He also outlines an account of applied mathematics intended to be capable of accounting for the applicability of inconsistent mathematics.

Christopher Pincock exemplifies the “naturalistic turn” in the philosophy of mathematics, which is characterized by a greater attention to actual mathematical practice. He argues that applied mathematics differs from pure mathematics in its priorities, methods, and standards of adequacy. These claims are illustrated by means of a case study, namely so-called “boundary layer theory”. He shows how in applied mathematics wholly mathematical justifications are not always available, as there is also a need to rely on experiment and observation. There may also be semantic and metaphysical differences between pure and applied mathematics based on these differences.

Part IV. Mathematical language and the psychology of mathematics

Many philosophers argue that a philosophical interpretation of mathematics must be sensitive to the way mathematical language is used and to the psychology of mathematical thought. This concern plays a prominent role in the contribution to Part IV of the volume.

Hofweber argues that progress in the philosophy of mathematics is hampered by excessive and misguided use of formal tools. He claims that the role for formal tools in the philosophy of mathematics is limited, at least as concerns the philosophical project of answering questions about truth, knowledge, and fact in mathematics. Hofweber points out that many questions in the philosophy of mathematics are empirical, for example, whether fictionalism is the correct interpretation of actual mathematical language and practice. He also warns that, although the language of Peano Arithmetic has very desirable mathematical properties (such simplicity and inferential adequacy), it does not provide an adequate linguistic analysis of actual arithmetical language. However, formal tools would have an important role to play in the philosophy of mathematics, Hofweber argues, if the axioms of certain parts of mathematics function constitutively rather than descriptively, while the axioms of arithmetic are descriptive rather than constitutive.

Øystein Linnebo discusses two competing accounts of how the natural numbers are individuated. His preferred notion of individuation is a semantic one which is concerned with our most fundamental ways of singling
out objects for reference in thought or in language. According to the cardinal account (defended by Fregeans and other logicians), a natural number is singled out by means of a concept or plurality which instantiates the number in question. According to the ordinal account (defended by structuralists and constructivists), a natural number is singled out by means of a numeral which occupies the corresponding position in a sequence of numerals. Linnebo criticizes the cardinal account for being linguistically and psychologically implausible. He then develops a version of the ordinal account which countenances numbers as objects but only in a very lightweight sense.

Agustín Rayo’s article defends the view that mathematical sentences have trivial truth-conditions; that is, that every mathematical truth is true in every intelligible scenario. This view would ensure that mathematics is epistemologically tractable. The notion of an intelligible scenario is explained in terms of two notions of identity: ordinary identity between individuals and an identity-like relation between properties (such as to be water just is to be H₂O). A sentence has trivial truth-conditions just in case it holds in every scenario compatible with all true identities. If trivialism is right and mathematical sentences have trivial truth-conditions, why is mathematical knowledge so valuable? Rayo answers that its value lies in an improved ability to distinguish intelligible from unintelligible scenarios.

Part V. From philosophical logic to the philosophy of mathematics

Finally, some contributors have approached the philosophy of mathematics using tools and methods from philosophical logic. Their work forms Part V of the collection.

Hannes Leitgeb studies the notion of informal provability, which he distinguishes from formal provability. The former notion is legitimate, Leitgeb argues, because it plays an important role in ordinary mathematical practice. Next he provides an interpretation and defense of some of Gödel’s cryptic remarks about the role of intuition in the epistemology of mathematics. This is based on a broadly naturalistic account of cognitive representations of mathematical structures. Leitgeb then turns to the logic of informal provability, agreeing with Gödel that this logic is at least S₄ and making some suggestions about how one may come to extend this logic even further. The chapter closes with some tentative suggestions about how we may come to establish that there are true but informally unprovable mathematical statements.

Gabriel Uzquiano discusses the threat which the phenomenon of “indefinite extensibility” poses to the existence of an all-comprehensive domain of quantification. The claim that there is an all-comprehensive domain of quantification is first distinguished from a strong form of metaphysical realism with which it is often conflated. A more neutral formulation of the claim, Uzquiano suggests, is that there are some objects such that every object is one of them. He then explores a challenge to this version of the claim. Assume any plurality of objects form a set. Then there cannot be an all-comprehensive domain; for if there were, there would be a universal set, which by Russell’s paradox would lead to a contradiction. Can the assumption that every plurality forms a set be incorporated into set theory? Uzquiano argues that this is possible only at the cost of denying some important features of contemporary set theory.

Concluding remarks

As this brief outline indicates, the contributions to this volume offer “new waves” to the philosophy of mathematics in two senses. Several contributions revisit established philosophical issues about mathematics but offer new ways of framing, conceptualizing, and approaching these issues. Other contributions raise new philosophical issues about mathematics and its practice that haven’t been addressed as forcefully as they should have. And several contributions do both. Since the chapters that follow speak for themselves, we leave it as an exercise to the reader to identify which contributions do what. We hope this may be an interesting way of enjoying the journey, while the waves will take you to new, exciting, and often rather unexpected places.

Note

1. Hellman’s modal-structural interpretation of mathematics was also inspired by an earlier view advanced by Putnam in his defense of “mathematics without foundations".