3 Mathematical Fictionalism

Olávio Bueno

In this chapter, I highlight five desiderata that an account of mathematics should meet to make sense of mathematical practice. After briefly indicating that current versions of platonism and nominalism fail to satisfy all of the desiderata, I sketch two versions of mathematical fictionalism that meet them. One version is based on an empiricist view of science, and has the additional benefit of providing a unified account of both mathematics and science. The other version of fictionalism is based on the metaphysics of fiction and articulates what can be considered a truly fictionalist account of mathematics. I indicate that both versions of fictionalism satisfy all of the desiderata, and I take it that they are best developed if adopted together. As a result, mathematical fictionalism is alive and well.

1 Introduction: Platonism and nominalism

Platonism is the view according to which there are abstract entities (such as sets, functions, and numbers), and mathematical theories truly describe such objects and the relations among them. Given the nature of mathematical entities—especially the fact that they are not located in space and time, and are causally inert—the postulation of such objects doesn’t come lightly. Platonists are, of course, well aware of this fact. It’s important, then, to highlight the benefits that immediately emerge from positing mathematical objects.

Three main benefits should be highlighted as follows:

(a) Mathematical discourse can be taken at face value (i.e. it can be taken literally), given that, according to the platonist, mathematical terms refer. So, when mathematicians claim that “There are infinitely many prime numbers”, the platonist can take that statement literally as describing the existence of an infinitude of primes. On the platonist view, there are obvious
truth makers for mathematical statements: mathematical objects and their corresponding properties.

This is a major benefit of platonism. If one of the goals of the philosophy of mathematics is to provide understanding of mathematics and mathematical practice, the fact that platonists are able to take the products of that practice—such as mathematical theories—literally and do not have to rewrite or reformulate them is a significant advantage. After all, the platonist is in a position to examine mathematical theories as they are actually formulated in mathematical practice, rather than discuss a parallel discourse offered by various reconstructions of mathematics given by those who avoid the commitment to mathematical objects (the nominalists).

(b) The platonist can also provide a unified semantics for both mathematical and scientific statements. Again, given the existence of mathematical objects, mathematical statements are true in the same way as scientific statements are. The only difference emerges from their respective truth makers: mathematical statements are true in virtue of abstract (mathematical) objects and the relations among the latter, whereas scientific statements are true, ultimately, in virtue of concrete objects and the corresponding relations among such objects.

Moreover, as is typical in the application of mathematics, there are also mixed statements, which involve terms referring to concrete objects and to abstract ones. The platonist has no trouble providing a unified semantics for such statements as well—particularly if platonism about mathematics is associated with realism about science. In this case, the platonist can provide a referential semantics throughout.

(c) Finally, platonists will also insist that it’s possible to explain the success of the application of mathematics, given that mathematical theories are taken to be true and mathematical terms refer to appropriate mathematical objects. So, it’s not surprising that mathematical theories can be used so successfully to describe relations among physical objects. True mathematical theories correctly describe relations among mathematical objects, and suitably interpreted, such relations are then used to account for various features in the physical world.

However, the platonist can explain the application of mathematics is actually controversial. Given that mathematical objects are abstract, it’s unclear why the postulation of such entities is helpful to understand the success of applied mathematics. After all, the physical world—being composed of objects located in space and time—is not constituted by entities of the same kind as those postulated by the platonist. Hence, it’s not clear why to describe correctly relations among abstract (mathematical) entities is even relevant to understand the behavior of concrete objects in the physical world involved in the application of mathematics. Just mentioning that the physical world instantiates structures (or substructures) described in general terms by various mathematical theories, for example, as in Shapiro (1997), is not enough. For there are infinitely many mathematical structures, and there’s no way of uniquely determining which of them is actually instantiated—or even instantiated only in part—in a finite region of the physical world. There’s a genuine underdetermination here, given that the same physical structure in the world can be accommodated by very different mathematical structures. For instance, according to Weyl (1928), quantum mechanical phenomena can be characterized in terms of group-theoretic structures, and according to von Neumann (1932), in terms of structures emerging from the theory of Hilbert spaces. Mathematically, such structures are very different, but there’s no way of deciding between them empirically. (I’ll return to this point below.) In any case, despite the controversial nature of the platonist claim, explaining the success of applied mathematics is often taken as a significant benefit of platonism.

Taken together, benefits (a)–(c) provide crucial components for platonists to make sense of mathematical practice, given that mathematical theories will not be rewritten for philosophical purposes, which allows for the many uses of mathematics, including its applications, to be understood in their own terms. Mathematical practice can be accommodated as is.

But platonism also has its costs: (a) Given that mathematical objects are causally inert and are not located in space and time, how exactly can mathematical knowledge be explained in the absence of any direct access to these objects? (b) For a similar reason, how exactly is reference to mathematical entities achieved? Platonists are, of course, well aware of the issue, and they have developed various strategies to address the problem. But it’s still contentious how successful these strategies turn out to be.

These costs motivate the development of an alternative view that doesn’t presuppose the commitment to mathematical entities. According to nominalism, there are no abstract entities, or at least, they are not required to make sense of mathematics and its applications.

Nominalism arguably has two main benefits: (a) Given that an ontology of mathematical entities is not presupposed, the possibility of mathematical knowledge is taken to be unproblematic. For instance, as proposed by Field (1989), mathematical knowledge is ultimately taken to be empirical or logical knowledge. (b) The same goes for reference to mathematical objects. Very roughly, if there are no mathematical entities, there is nothing there to be referred to! The issue simply vanishes.

But nominalism, just as platonism, also has its costs: (a) Mathematical discourse is not taken at face value (the discourse is not taken literally). Each nominalization strategy for mathematics introduces some change in either the syntax or the semantics of mathematical statements. In some cases, modal operators are introduced to preserve verbal agreement with the platonist, as in the studies by Hellman (1989). The proposal is that each mathematical statement S is translated into two modal statements: (i) If there were structures of the suitable kind, S would be true in these structures, and
(ii) It’s possible that there are such structures. As a result, both the syntax and the semantics of mathematics are changed. In other cases, as can be seen in Field (1989), in order to preserve verbal agreement with the platonist despite the negation that mathematical objects exist, fiction operators (such as: “According to arithmetic...”) are introduced. Once again, the resulting proposal shifts the syntax and the semantics of mathematical statements.

(b) Given that mathematical statements are not taken literally, it comes as no surprise that, on nominalist views, the semantics of science and mathematics is not uniform. After all, modal and fiction operators need to be introduced. But these operators have no counterparts in science given that scientific theories are taken to provide descriptions of the world, and the presence of these operators prevents us from making any such claims.

Finally, (c) it is not clear that the nominalist can provide an account of the application of mathematics. After all, if mathematical terms do not refer and mathematical theories are not true, why is it that these theories are so successful in science? Prima facie, it becomes mysterious exactly why that success should emerge, given the nonexistence of mathematical objects. Nominalists are, of course, aware of the issue, and some try to offer a story as to why mathematical theories can be used successfully despite being false. According to Field (1980, 1989), mathematical theories need not be true to be good, as long as they are conservative, that is, consistent with every internally consistent claim about the physical world. Field then argues that, given the conservativeness of mathematics, it’s possible to dispense with mathematical objects in derivations of claims from nominalistic premises to nominalistic conclusions (that is, in claims in which mathematical terms do not occur). However, this move fails to explain the success of mathematical theories as the latter are in fact applied in mathematical practice. The fact that the nominalistic versions of certain physical theories may work in application doesn’t explain how actual mathematical theories manage to work. Once again, we have an interesting philosophical proposal that produces a parallel discourse, and in terms of this discourse a particular ontological claim is made about actual practice; namely, that it need not be committed to mathematical objects, given the dispensability of the latter. But this leaves entirely open the issue as to which features in actual mathematical practice, if any, could be invoked to avoid commitment to mathematical objects, given that scientists don’t formulate their theories in accordance with Field’s nominalistic recipe.

Given (a)–(c), the nominalist is ultimately unable to accommodate mathematical practice. Given that mathematical statements need to be rewritten, the practice cannot be taken literally. Moreover, if actual cases of the application of mathematics are not accommodated, a significant dimension of mathematical practice is left unaccounted for. An alternative proposal is thus required.

2 Motivations for fictionalism

The considerations above motivate the following question: Is it possible to develop a view that has all the benefits of platonism without the corresponding costs? Or, equivalently, a view that has none of the costs of nominalism, while keeping all of its benefits? In other words, what we need is a view that meets the following desiderata (each of which is independently plausible):

1. The view explains the possibility of mathematical knowledge.
2. It explains how reference to mathematical entities is achieved.
3. It accommodates the application of mathematics to science.
4. It provides a uniform semantics for mathematics and science.
5. It takes mathematical discourse literally.

Note that if all of these desiderata are met, the resulting view will be able to accommodate mathematical practice. After all, given that mathematical theories are taken literally, and a uniform semantics for mathematics and science is offered, there is no need for making up a parallel discourse as a replacement for the actual practice. With the desiderata in place, the resources are available to make sense of the practice in its own terms.

In this chapter, I argue that the desiderata above can all be met as long as a fictionalist view of mathematics is articulated, and I will sketch such a view. In fact, I’ll sketch two such views, putting forward two different strategies to articulate a fictionalist stance, and indicating the ways in which the strategies support each other. For obvious reasons, I’ll only be able to offer an outline of the views here. But hopefully enough will be said to indicate how the views look like.

3 Two fictionalist strategies

3.1 Fictionalism and nominalism

What is the difference between fictionalism and nominalism? As developed here, fictionalism is an agnostic view; it doesn’t state that mathematical objects don’t exist. Rather, the issue of their existence is left open. Perhaps these objects exist, perhaps they don’t. But, according to the fictionalist, we need not settle the issue to make sense of mathematics and mathematical practice. Thus, in contrast with the skeptical view offered by the nominalist—who denies the existence of mathematical objects and relations—the fictionalist provides an agnostic proposal. In contrast with platonism, the fictionalist is not committed to the existence of mathematical objects and relations either. In this way, and at least in temperament, fictionalism is closer to nominalism than to platonism. In fact, to highlight the connection with nominalism, the fictionalist view can also be called
agnostic nominalism. After all, fictionalism provides a strategy to avoid commitment to the existence of mathematical entities, but without, thereby, denying their existence.

The two strategies sketched below are fictionalist in slightly different ways. The first strategy assumes a particular empiricist view about science—namely, constructive empiricism, as developed by van Fraassen (1980, 1989)—and indicates how to extend this view to make sense of mathematics. This strategy is fictionalist in a broad sense; it offers a way of using mathematics that is compatible with a fictionalist reading of mathematical statements. The second strategy explores the metaphysics of fiction, and what it takes to introduce a fictional object. This strategy is fictionalist in a narrow sense; it indicates directly the similarities between mathematical and fictional entities.

3.2 The empiricist fictionalist strategy

The empiricist strategy focuses on applied mathematics, which is the central feature of mathematics that an empiricist needs to make sense of as part of his or her account of science. Given the restriction to applied mathematics, the empiricist strategy will not offer a general account of mathematics. The second, truly fictionalist strategy, is more general—also in that it doesn’t presuppose a commitment to an empiricist view of science. In turn, the second strategy doesn’t have anything special to say about science. In the end, the two views are better adopted together.

3.2.1 The crucial idea

According to constructive empiricism, the aim of science is not truth, but something weaker. As van Fraassen (1980) suggests, it is empirical adequacy. Roughly, a theory is empirically adequate if it’s true about the observable features of the world. As for the unobservable features, the constructive empiricist remains agnostic: it’s not clear how we can know whether unobservable entities exist or not, and in case they do exist, which features they have. After all, there are incompatible accounts of unobservable phenomena that characterize the latter in drastically different ways, but which turn out to be all empirically adequate. Consider, for instance, Copenhagen and Bohmian interpretations of quantum mechanics. According to the Copenhagen view, quantum objects are such that it’s not possible to measure simultaneously their position and momentum with full certainty. In contrast, according to the Bohmian conception, quantum objects can be so measured. Both interpretations are empirically adequate, and thus cannot be chosen based on empirical consideration alone. But they offer strikingly different accounts of the nature of quantum objects. Given that, empirically, we cannot choose between them, it’s unclear that, in the end, we are in a position to determine the nature of quantum objects. This is, of course, a familiar underdetermination argument, and the empiricist explores it to motivate agnosticism about unobservable phenomena. So, the constructive empiricist neither denies nor asserts the existence of unobservable entities. What is offered is an agnostic stance.

The crucial idea of the empiricist fictionalist strategy to applied mathematics is to insist that (applied) mathematical theories need not be true to be good. They only need to be part of an empirically adequate package. This theoretical package typically involves: a scientific theory, the relevant mathematical theories (used in the formulation of the scientific theory in question), interpretations of the resulting formalism, and initial conditions. The whole package is never asserted to be true; it’s only required to be empirically adequate, that is, to accommodate the observable phenomena.

Now, as noted, empirical adequacy is weaker than truth—it’s truth about the observable phenomena. In particular, a theory’s empirical adequacy doesn’t establish the existence of unobservable objects—whether mathematical or physical. Given that the empirical adequacy of a theoretical package is compatible with this package being mistaken in its description of the unobservable phenomena—which includes reference to both physical and mathematical objects—the empiricist is not committed to unobservable objects when an empirically adequate package is adopted. In the end, the existence of unobservable objects is not required to make sense of scientific or mathematical practice. As a result, unobservable objects can be taken as fictional.

Similarly to what the constructive empiricist does in the context of science, the fictionalist can use underdetermination arguments to motivate agnosticism about the existence of mathematical objects. After all, it’s possible to obtain the same empirical consequences of a given scientific theory using significantly different mathematical frameworks. For example, quantum mechanics can be formulated via group theory, as Weyl (1928) proposed, or via Hilbert spaces, as von Neumann (1932) did. Mathematically, these are very different formulations, which emphasize different aspects of the quantum mechanical formalism. Weyl was particularly interested in characterizing some features of quantum objects, and the use of group theory, with its transformation groups, was central to this task. In turn, von Neumann was especially concerned with offering a systematic framework to represent quantum states and introduce probability into quantum mechanics. The Hilbert space formalism was appropriate for both tasks. Despite the significant mathematical differences between the two frameworks, which emerge from the different mathematical theories that are presupposed in each case, the same empirical results about quantum phenomena are obtained.

The empiricist fictionalist will then note that the underdetermination of these two theoretical packages motivates agnosticism about the mathematical objects that are invoked in the mathematical formulation of quantum
mechanics. Should the empiricist be committed to the existence of group-theoretic transformations given the success of the application of group theory to quantum mechanics? Or should the commitment go for vectors in a Hilbert space instead, given the success of the corresponding theory in quantum mechanics? Recall that each package offers a different account of what is going on beyond the observable phenomena. And given their empirical equivalence, it's unclear how to choose between them on empirical grounds. Agnosticism then emerges.

It might be argued that the empiricist should be committed to the existence of the two types of objects, given that groups and vectors have been both successfully applied in quantum mechanics. The trouble with this suggestion, however, is that we don't get a coherent picture from the adoption of both mathematical frameworks. Different features of quantum objects and their states are articulated in each case. Even if we adopted both frameworks, it wouldn't still be clear what the content of the resulting package is supposed to be. Once again, to remain agnostic seems to be the warranted option in this case.

It might be objected that the notion of empirical adequacy that is invoked here is not available to the empiricist fictionalist. After all, as formulated by van Fraassen (1980, p. 64), the concept of empirical adequacy presupposes abstract entities. It is, thus, a concept that the empiricist fictionalist should be agnostic about. As formulated by van Fraassen, a theory is empirically adequate if there is a model of that theory such that every empirical substructure of that model is isomorphic to the appearances (i.e., the structures that represent the outcomes of the experimental results). Given that the models involved are themselves mathematical objects, when an empiricist believes that a scientific theory is empirically adequate, she will thereby believe in the existence of abstract entities, as Rosen (1994) argues.

In response, two moves are available for the empiricist fictionalist. First, she can adopt a formulation of empirical adequacy that does not presuppose abstract entities; for instance, the characterization according to which a theory is empirically adequate as long as what it states about the observable phenomena is true, as presented in van Fraassen (1980). Second, the empiricist can use the (truly) fictionalist strategy developed below to accommodate this difficulty. After all, as will become clear shortly, the truly fictionalist strategy has the resources to accommodate structures from pure mathematics, such as mathematical models and transformations among them.

3.2.2 Meeting the desiderata

Can the empiricist fictionalist strategy accommodate the five desiderata discussed above? I think it can. Here—in very broad outline—is how this can be done.

1. Mathematical knowledge. Can we expect to get any account of mathematical knowledge from a proposal that is agnostic about the existence of mathematical objects? Certainly, the proposal won't yield knowledge of the objects that, according to the platonist, make mathematical statements true. For these are precisely the objects about which the empiricist fictionalist is agnostic. So, what exactly can be expected in this case?

   Consider the corresponding constructive empiricist view about knowledge of unobservables in science. Given the constructive empiricist's agnosticism about unobservable phenomena, it comes as no surprise that the empiricist doesn't claim to know what is going on at the unobservable level. He or she suspends the judgment about the issue. There is, however, a more positive component to the constructive empiricist approach to the problem. The fact that many incompatible, but empirically adequate, accounts of unobservable phenomena are available—such as the different interpretations of quantum mechanics—provides an important form of understanding, namely, of how the world could be if these underdetermined accounts or interpretations were true, as van Fraassen (1991, 1989) points out. Each interpretation indicates a possible way the unobservable phenomena behave in order to generate the observable features of the world that we do experience. Although we may not be able to decide which of these interpretations (if any) is true, we can still understand the conception of the unobservable world that each of them provide. For example, if Copenhagen-type quantum objects populate the world, we can understand why the position and momentum of these objects cannot be simultaneously measured with full certainty. If, however, in accordance with the Bohmian interpretation, a quantum potential exists, we can then make sense of how the position and momentum of quantum objects can be measured simultaneously. In each case, we understand how the world could be, even if we don't know how it actually is.

   Similarly, in the case of the mathematical theories used in theoretical packages, it becomes clear that each of them gives us understanding. We understand how central features of quantum particles can be expressed by formulating quantum mechanics in terms of group-theoretic invariants. We can understand how quantum states can be represented in terms of suitable features of a Hilbert space. In the end, we understand how the world could be if the theoretical packages were true, despite the fact that we are unable to determine which of them (if any) is true, as van Fraassen (1991) emphasizes. And the fact that we are genuinely unable to choose between the theoretical packages on empirical grounds helps us understand why being agnostic about the existence of the corresponding objects is a perfectly acceptable stance.

   Thus, in the empiricist fictionalist picture, mathematical knowledge becomes part of scientific knowledge—at least with regard to applied mathematics. We get to know mathematical results, in part, by understanding the
role they play in the investigation of the empirical world. Of course, the concept of knowledge at work here is very deflationary since only the truth of the observable aspects of the theories in question is involved. This includes the observable components of applied mathematical theories. In this respect, the limited knowledge involved in applied mathematics is similar to the corresponding knowledge that the constructive empiricist recognizes in science. Whether in science or applied mathematics, knowledge and understanding go hand in hand.

(2) Reference to mathematical entities: How do we refer to mathematical entities on the empiricist fictionalist strategy? We refer to them in exactly the same way as we refer to unobservable objects in science. In fact, "electrons" refer to electrons just as "sets" refer to sets. Recall that the empiricist does not deny the existence of unobservable entities, whether they are mathematical or empirical. Rather than a skeptical attitude about these entities, what is offered is an agnostic one. To refer to mathematical objects or to unobservable physical entities, such as quarks or photons, all the empiricist needs is a theory that characterizes some properties of the relevant objects, even though he or she may be agnostic about whether these entities exist or not.

Although reference is often used as a success term, particularly in philosophical contexts, there is no need to assume that this is the case. After all, clearly we can refer to non-existing things, such as fictional characters (e.g. Sherlock Holmes) and non-existing posits of scientific theorizing (e.g. phlogiston). The fact that, in these cases, the corresponding objects do not exist doesn't prevent us from referring to them. We do that all the time. In the end, as we can see in Azzouni (2004), reference need not require existence of the objects that are referred to.

Thus, the mechanism of reference to mathematical entities is not different from the one in terms of which we refer to unobservable entities in science. The main difference is that, although there are mechanisms of instrumental access to scientific entities, there aren't such mechanisms in the case of mathematical objects. This is how it should be, since we don't expect the existence of mechanisms of instrumental access to abstract entities, given that the latter are not located in space and time.

(3) Application of mathematics: Applied mathematics is often used as a source of support for platonism. How else but by becoming platonists can we make sense of the success of applied mathematics in science? As an answer to this question, the fictionalist empiricist will note that it's not the case that applied mathematics always works. In several cases, it doesn't work as initially intended, and it works only when accompanied by suitable empirical interpretations of the mathematical formalism. For example, when Dirac found negative energy solutions to the equation that now bears his name, he tried to devise physically meaningful interpretations of these solutions. His first inclination was to ignore these negative energy solutions as not being physically significant, and he took the solutions to be just an artifact of the mathematics—as is commonly done in similar cases in classical mechanics. Later, however, he identified a physically meaningful interpretation of these negative energy solutions in terms of "holes" in a sea of electrons. But the resulting interpretation was empirically inadequate, since it entailed that protons and electrons had the same mass. Given this difficulty, Dirac rejected that interpretation and formulated another. He interpreted the negative energy solutions in terms of a new particle that had the same mass as the electron but opposite charge. A couple of years after Dirac's final interpretation was published Anderson detected something that could be interpreted as the particle that Dirac postulated. Asked as to whether Anderson was aware of Dirac's papers, Anderson replied that he knew of the work, but he was so busy with his instruments that, as far as he was concerned, the discovery of the positron was entirely accidental. For further details and references, see Bueno (2005).

The application of mathematics is ultimately a matter of using the vocabulary of mathematical theories to express relations among physical entities. Given that, for the fictionalist empiricist, the truth of the various theories involved—mathematical, physical, biological, and whatnot—is never asserted, no commitment to the existence of the entities that are posited by such theories is forthcoming. But if the theories in question—and, in particular, the mathematical theories—are not taken to be true, how can they be successfully applied? There is no mystery here. First, even in science, false theories can have true consequences. The situation here is analogous to what happens in fiction. Novels can, and often do, provide insightful, illuminating descriptions of phenomena of various kinds—for example, psychological or historical events—that help us understand the events in question in new, unexpected ways, despite the fact that the novels in question are not true. Second, given that mathematical entities are not subject to spatial-temporal constraints, it's not surprising that they have no active role in applied contexts. Mathematical theories need only provide a framework that, suitably interpreted, can be used to describe the behavior of various types of phenomena—whether the latter are physical, chemical, biological, or whatnot. Having such a descriptive function is clearly compatible with the (interpreted) mathematical framework not being true, as Dirac's case illustrates so powerfully. After all, as was just noted, one of the interpretations of the mathematical formalism was empirically inadequate.

(4) Uniform semantics: On the fictionalist empiricist account, mathematical discourse is clearly taken on a par with scientific discourse. There is no change in the semantics. Mathematical and scientific statements are treated
in exactly the same way. Both sorts of statements are truth-apt, and are taken as describing (correctly or not) the objects and relations they are about. The only shift here is on the aim of the research. After all, on the fictionalist empiricist proposal, the goal is not truth, but something weaker: empirical adequacy—or truth only with respect to the observable phenomena. However, once again, this goal matters to both science and (applied) mathematics, and the semantic uniformity between the two fields is still preserved.

(5) **Taking mathematical discourse literally:** According to the fictionalist empiricist, mathematical discourse is also taken literally. If a mathematical theory states that “There are differentiable functions such that...”, the theory is not going to be reformulated in any way to avoid reference to these functions. The truth of the theory, however, is never asserted. There's no need for that, given that only the empirical adequacy of the overall theoretical package is required.

### 3.3 The fictionalist strategy

Although the two strategies described in this chapter are fictionalist, the second strategy is truly fictionalist in the sense that it explores fictionalism from the metaphysics of fiction. To distinguish these strategies, from now on I’ll call the first the **empiricist strategy**, and the second the **fictionalist strategy**.

#### 3.3.1 The crucial point

The central point of the fictionalist strategy is to emphasize that mathematical entities are like fictional entities. They have similar features that fictional objects such as Sherlock Holmes or Hamlet have. By indicating that there is nothing mysterious in the way in which we can have knowledge of fictional entities and are able to refer to them, and by arguing that mathematical entities are a particular kind of fictional entity, a truly fictionalist view can be articulated.

The fictionalist’s proposal is to extend to mathematics the work on the nature of fictional characters that Amie Thomasson has developed (1999). Thomasson put forward the artifact theory of fictional objects, according to which the latter objects are abstract artifacts. First, fictional objects are created by the intentional acts of their authors (in this sense, they are artifacts). So, they are introduced in a particular context, in a particular time. Second, fictional objects depend on (i) the existence of copies of the artworks that describe these objects (or through memories of such works), and (ii) the existence of a community who is able to understand these works. In other words, fictional objects depend on the existence of concrete objects in the physical world (e.g., books, a community of readers, etc.). As a result, there is nothing mysterious about the way in which we refer to and obtain knowledge of such objects. By reading the story, we get information about the entities in question, and get a sense of what happened to them.

Similar points apply to mathematical entities. First, these entities are also created, in a particular context, in a particular time. They are artifacts. Mathematical entities are created when comprehension principles are put forward to describe their behavior, and when consequences are drawn from such principles. Second, mathematical entities thus introduced are also dependent on (i) the existence of particular copies of the works in which such comprehension principles have been presented (or memories of these works), and (ii) the existence of a community who is able to understand these works. It's a perfectly fine way to describe the mathematics of a particular community as being lost if all the copies of their mathematical works have been lost and there's no memory of them. Thus, mathematical entities, introduced via the relevant comprehension principles, turn out to be contingent—at least in the sense that they depend on the existence of particular concrete objects in the world, such as, suitable mathematical works. The practice of mathematics depends crucially on producing and referring to these works.

In this respect, the fictionalist insists that there is nothing mysterious about how we can refer to mathematical objects and have knowledge of them. Similarly to the case of fiction, reference to mathematical objects is made possible by the works in which the relevant comprehension principles are formulated. In these works, via the relevant principles, the corresponding mathematical objects are introduced. The principles specify the meaning of the mathematical terms that are introduced as well as the properties that the mathematical objects that are thus posited have. In this sense, the comprehension principles provide the context in which we can refer to and describe the mathematical objects in question. Our knowledge of mathematical objects is then obtained by examining the properties that these objects have, and by drawing consequences from the comprehension principles.

#### 3.3.2 On the existence of mathematical objects

What about the mathematical objects that, according to the platonist, exist independently of any description one may offer of them in terms of comprehension principles? Do these objects exist on the fictionalist view? Now, the fictionalist is not committed to the existence of such mathematical objects, although this doesn't mean that the fictionalist is committed to the non-existence of these objects. On the view advanced here, the fictionalist is ultimately agnostic about the issue. Here is why.

According to Azzouni (1997b, 2004), there are two types of commitment: quantifier commitment and ontological commitment. We incur quantifier commitment to the objects that are in the range of our quantifiers. We incur
ontological commitment when we are committed to the existence of certain objects. However, despite Quine's view, quantifier commitment doesn't entail ontological commitment. Fictional discourse (e.g. in literature) and mathematical discourse illustrate that. Suppose that there's no way of making sense of our practice with fiction but to quantify over fictional objects, such as Sherlock Holmes or Pegasus. Still, people would strongly resist the claim that they are therefore committed to the existence of these objects. The same point applies to mathematical objects.

This move can also be made by invoking a distinction between partial quantifiers and the existence predicate. The idea here is to resist reading the existential quantifier as carrying any ontological commitment. Rather, the existential quantifier only indicates that the objects that fall under a concept (or have certain properties) are less than the whole domain of discourse. To indicate that the whole domain is invoked (e.g. that every object in the domain have a certain property), we use a universal quantifier. So, two different functions are clumped together in the traditional, Quinean reading of the existential quantifier: (i) to assert the existence of something, on the one hand, and (ii) to indicate that not the whole domain of quantification is considered, on the other. These functions are best kept apart. We should use a partial quantifier (that is, an existential quantifier free of ontological commitment) to convey that only some of the objects in the domain are referred to, and introduce an existence predicate in the language in order to express existence claims.

By distinguishing these two roles of the quantifier, we also gain expressive resources. Consider, for instance, the sentence:

(*) Some fictional detectives don't exist.

Can this expression be translated in the usual formalism of classical first-order logic with the Quinean interpretation of the existential quantifier? Prima facie, that doesn't seem to be possible. The sentence would be contradictory! It would state that there exist fictional detectives who don't exist. The obvious consistent translation here would be: ¬∃x Fx, where F is the predicate is a fictional detective. But this states that fictional detectives don't exist. Clearly, this is a different claim from the one expressed in (*). By declaring that some fictional detectives don't exist, (*) is still compatible with the existence of some fictional detectives. The regimented sentence denies this possibility.

However, it's perfectly straightforward to express (*) using the resources of partial quantification and the existence predicate. Suppose that "∃" stands for the partial quantifier and "E" stands for the existence predicate. In this case, we have: ∃x (Fx ∧¬Ex), which expresses precisely what we need to state.

Now, under what conditions is the fictionalist entitled to conclude that certain objects exist? In order to avoid begging the question against the platonist, the fictionalist cannot insist that only objects that we can causally interact with exist. So, the fictionalist only offers sufficient conditions for us to be entitled to conclude that certain objects exist. Conditions such as the following seem to be uncontroversial. Suppose we have access to certain objects that is such that (i) it's robust (e.g. we blink, we move away, and the objects are still there); (ii) the access to these objects can be refined (e.g. we can get closer for a better look); (iii) the access allows us to track the objects in space and time; and (iv) the access is such that if the objects weren't there, we wouldn't believe that they were. In this case, having this form of access to these objects gives us good grounds to claim that these objects exist. In fact, it's in virtue of conditions of this sort that we believe that tables, chairs, and so many observable entities exist.

But recall that these are only sufficient, and not necessary, conditions. Thus, the resulting view turns out to be agnostic about the existence of the mathematical entities the platonist takes to exist—indeed independently of any description. The fact that mathematical objects fail to satisfy some of these conditions doesn't entail that these objects don't exist. Perhaps these entities do exist after all; perhaps they don't. What matters for the fictionalist is that it's possible to make sense of significant features of mathematics without settling this issue.

Now what would happen if the agnostic fictionalist used the partial quantifier in the context of comprehension principles? Suppose that a vector space is introduced via suitable principles, and that we establish that there are vectors satisfying certain conditions. Would this entail that we are now committed to the existence of these vectors? It would if the vectors in question satisfied the existence predicate. Otherwise, the issue would remain open, given that the existence predicate only provides sufficient, but not necessary, conditions for us to believe that the vectors in question exist. As a result, the fictionalist would then remain agnostic about the existence of even the objects introduced via comprehension principles!

3.3.3 Meeting the desiderata

Does the fictionalist's account satisfy the five desiderata discussed in Section 2? I think it does. In very broad outline, here is why.

1. Mathematical knowledge: Knowledge of mathematical entities, just as knowledge of fictional entities in general, is the result of producing suitable descriptions of the objects in question and drawing consequences from the assumptions that are made. Central in this process, we saw, is the formulation of comprehension principles, which specify and systematize the use of the relevant mathematical concepts. Note that once certain comprehension principles are introduced and a logic is adopted, it's no longer
up to us what follows from such principles. It's a matter of the relations that hold among the concepts that are introduced in the relevant comprehension principles as well as any additional assumptions that are invoked and the deductive patterns of inference that are used. In this sense, mathematical knowledge is objective, even though the objectivity in question does not depend on the existence of mathematical objects independently of comprehension principles.

The fictionalist acknowledges, of course, that we may not be able to establish some results based on certain comprehension principles; for example, the system in question may be incomplete. We may even prove that the system in question is incomplete—again, by invoking suitable comprehension principles and Gödel’s incompleteness theorems. Does this mean that we can determine the truth of certain mathematical statements (say, a particular Gödel sentence) independently of comprehension principles? It's not clear that this is the case. After all, the comprehension principles will be needed to characterize the meaning of the mathematical terms in question; they are part of the context that specifies the concepts under consideration, as well as the objects and their relations.

(2) Reference to mathematical entities: How is reference to mathematical objects accommodated in the fictionalist’s approach? Once again, comprehension principles play a central role here. By introducing particular comprehension principles, reference to mathematical objects is made possible. After all, the principles specify some of the properties that the objects that are introduced have, and by invoking these properties, it's possible to refer to the objects in question as those objects that have the corresponding properties. In other words, with the comprehension principles in place, it's typically unproblematic to secure reference to mathematical objects. The latter are identified as the objects that have the relevant properties.

Although this suggestion may work in most cases, some difficulties need to be addressed. Suppose, for instance, that the comprehension principles don't uniquely determine the objects that are referred to. In this case, reference won't be sharp, in the sense that exactly one object is picked out. The best we can do is secure reference to an equivalence class of objects, without uniquely identifying each member of the class. In mathematical contexts, this is typically enough, given that mathematical structures are often characterized “up to isomorphism”. In other cases, the comprehension principles may turn out to be inconsistent. As a result, there won't be any consistent objects to refer to. In cases of this sort, we can still refer to inconsistent objects. This is acceptable to the fictionalist, who is not committed, of course, to the existence of such objects. We can refer, and we often do refer, to non-existent things. They are, ultimately, objects of thought.

In other words, similarly to knowledge of mathematical entities, reference to mathematical objects is achieved by invoking the relevant comprehension principles. As a result, reference is always contextual: it’s made in the context of the comprehension principles that give meaning to the relevant mathematical terms.

(3) Application of mathematics: The fictionalist can make sense of crucial features of the application of mathematics, while remaining agnostic about the existence of mathematical entities. The key idea is that, for the fictionalist, the application of mathematics is a matter of using the expressive resources of mathematical theories to accommodate different aspects of scientific discourse. But this doesn’t require the truth of the relevant mathematical theories. After all, whether or not mathematical theories correctly describe independently existing mathematical objects and their relations is largely irrelevant to the application of mathematics. What matters are the relations that are introduced by the mathematical theories and whether these relations can be interpreted in a physically significant way. And these issues are independent of the truth of the mathematical theories in question.

However, this still leaves the question as to why positing fictional objects can be so useful in the description of the physical world. The answer to this question depends, again, on the context. In some contexts, mathematical theories can be extremely useful. For instance, if we are interested in capturing certain structural properties of empirical phenomena—such as the representation of their speed, momentum, acceleration, or rate of growth—the mathematical vocabulary offers a rich, nuanced framework. In other contexts, however, mathematical theories are far from useful. For instance, suppose that we are interested in capturing the psychological states of some empirical phenomena; in this case, it’s unclear that, in general, the use of mathematical vocabulary is of much relevance. The same point applies to fictional discourse. Novels often offer insightful accounts of human psychology, but they clearly are inadequate sources of information regarding the representation of quantum states. In order to make sense of the application of mathematics, it’s crucial that we are sensitive to the context in which the mathematics is in fact used.

(4) Uniform semantics: On the fictionalist’s proposal, there is uniform semantics for scientific and mathematical statements. On this view, mathematical and scientific terms are treated in exactly the same way. There is no attempt to offer special semantic conditions for mathematical statements. Once the concept of prime number is introduced, the following statement: “There are infinitely many prime numbers” is true as long as there are infinitely many prime numbers. Of course, the existence of prime numbers is left open given the agnostic nature of the fictionalist view. But this is no change in the semantics for mathematical statements.

(5) Taking mathematical discourse literally: On the fictionalist proposal, mathematical discourse is also taken literally, given that the semantics of mathematical statements has not changed. The statement “There are infinitely many prime numbers” comes out true in the context, say, of
In this context, reference to prime numbers is achieved by the comprehension principles that provide meaning to the relevant terms.

It might be objected that the fictionalist is not taking mathematical discourse literally. Typically, fictionalists about mathematics (say, arithmetic) introduce a fiction operator “According to (arithmetic)”. So, the syntax of mathematical statements has to be tinkered with, and as a result, mathematical discourse is not taken literally.

In response, note that the practice of mathematics always presupposes a given mathematical “theory”. Certain mathematical principles (that need not be axiomatized) from which mathematicians draw their consequences. The results are always established based on such principles that provide the context for mathematical research. The fictionalist is not introducing a fiction operator to mathematical statements. The statements are used in the context of principles that characterize the properties of the relevant mathematical objects. In this sense, the fiction operator—in the form of the comprehension principles that specify a certain domain of objects—is already in place as part of mathematical practice. The fictionalist is not adding a new item to the language of mathematics. Properly conceptualized, the fiction operator is already there.

4 Conclusion

There is, of course, much more to be said about the two fictionalist strategies that were developed here, and I plan to expand on them in future work. My goal here has been only to sketch some of the central ideas of these proposals. If they are near the mark, it’s indeed possible to be fictionalist about mathematics—making sense, in particular, of mathematical practice along the way.

Notes

1. I’d like to thank my colleagues in the Philosophy Department at the University of Miami, the participants of the New Waves in Philosophy of Mathematics Conference, and Jody Azzouni for their extremely helpful responses to earlier versions of this work.

2. For different ways of formulating platonism, see, for instance, Quine (1960), Resnik (1997), Shapiro (1997), the first part of Balaguer (1998), and Colyvan (2001).

3. This point is significantly idealized in that it assumes that, somehow, we can manage to distill the empirical content of scientific statements independently of the contribution made by the mathematics that is often used to express such statements. According to Quine (1960) and Colyvan (2001), platonists who defend the indispensability argument will insist that this is not possible to do.

4. Of course, the platonist about mathematics need not be a realist about science—although it’s common to combine platonism and realism in this way. In principle, the platonist could adopt some form of anti-realism about science, for instance, constructive empiricism, as developed by van Fraassen (1980). As long as the form of anti-realism regarding science allows for a referential semantics (and many do), the platonist would have no trouble providing a unified semantics for both mathematics and science.

5. See the first part of Balaguer (1998) for a critical assessment of these strategies.

6. Different versions of nominalism can be found, for example, in Field (1980, 1989), Hellman (1989), Chihara (1990), Burgess and Rosen (1997), the second part of Balaguer (1998), and Azzouni (2004). Of these views, Azzouni’s is the one that comes closest to meeting all of the desiderata discussed below. But it’s unclear to me that his view in fact meets them, although the point is controversial further details are discussed in Bueno and Zalta (2005) and Azzouni (2009).

7. Field, of course, doesn’t claim that mathematical theories should be formulated in accordance with his program. He is only pointing out that, as opposed to what the platonist argues based on the indispensability argument, the successful use of mathematics in science need not force us to be committed to the existence of mathematical objects. But, as a result, the issue of making sense of the way in which actual mathematical theories are applied in science is not addressed.

8. It might be argued that we should not take the uniformity of semantics as a desideratum for a philosophical account of mathematics. After all, presumably on a more traditional, non-Quinean approach, mathematics would be taken as substantially different from empirical science—in particular, given its a priori character. Thus, mathematics may well correctly demand a different semantics (as well as a different epistemology). So, the uniformity of the semantics may be too contentious and substantial to be taken as a requirement here.

There’s no doubt that mathematics and empirical science are importantly different. And the fact that the former is often taken to be a priori may well highlight a significant difference between the two domains. However, this doesn’t undermine the point that the adoption of a uniform semantics would be a benefit since it won’t demand a special semantic treatment for special fields. Having a uniform semantics would also simplify considerably the account of applied mathematics, given that the treatment of mixed statements—that include reference to both mathematical and non-mathematical objects—would be the same throughout applied contexts.

Moreover, the difference in epistemological status between mathematical and empirical claims, on its own, isn’t sufficient to justify changing the semantics of these statements. Consider, for instance, the case of modal realism—as in Lewis (1986). For the modal realist, modal knowledge (that is, knowledge of what is possible or necessary) is a priori, just as mathematical knowledge is. However, the semantics for modal and mathematical statements is the same. The only difference is that, in the case of modal statements, possible worlds make them true, whereas in the mathematical case, mathematical objects and relations make the corresponding statements true. Incidentally, on the modal realist account, not even the semantics of empirical statements would be different: such statements are true in virtue of the features of the actual world. In the end, independently of the epistemological story one offers, the possibility of providing a uniform semantics should be taken as a benefit.

9. “Fictionalism” is used in dramatically different ways in the literature. In this chapter, I’ll be using the expression in terms of the two strategies discussed below. I need a term that is somewhat neutral between nominalism and platonism,
but which indicates a view that is still closer to the former than to the latter. Fictionalism seems appropriate for that. So, I'll stick to it.

10. For further details, see Azzouni (1997a, 2004).

11. Dirac’s work here clearly illustrates the point that the same mathematical formalism is compatible with multiple physically significant interpretations. Some of these interpretations, as we saw in the case of Dirac’s first interpretation, may turn out to be empirically inadequate. So, the empirical success of the theoretical package involving the mathematical formalism, the relevant physical theories, and the corresponding interpretations cannot be attributed to the formalism alone. In fact, on its own, the formalism is compatible with virtually any physical outcome. Typically, at most some cardinality constraints about the size of the domain of interpretation are imposed by the mathematical formalism alone.

12. The comprehension principles in question need not be consistent. The only requirement is that they are not trivial; that is, not everything follows from them.

13. In the fiction case (e.g., in literary works), fictional objects are typically introduced simply by specifying the properties that the corresponding objects have in a much more informal way than in mathematics—even though the latter is practiced very informally too in comparison with the standards of mathematical logicians.

14. For further reference, see McGinn (2000).

15. These conditions are based on the work that Jody Azzouni has done on this issue (2004). But he put them to a very different use!

16. However, mathematical vocabulary can be extremely useful if we are interested in capturing quantitative aspects of psychological phenomena, as mathematical models used in psychology clearly show.

17. In many cases, to describe mathematical practice as presupposing a theory about a certain domain suggests more organization than what is actually found. Particularly in emerging mathematical fields (say, the first studies of complex numbers), the principles may be just rough attempts to systematize and describe a certain group of mathematical phenomena. Although rough, the principles are still a central feature of the practice.

References


