Paraconsistent Logic

NEWTON C. A. DA COSTA AND OTÁVIO BUENO

Introduction

In the past two centuries, logic progressed so much that it became a new domain of inquiry. It is a formal and abstract discipline like mathematics. In fact, it is difficult to separate the fields of logic and mathematics, at least as they are developed in the contemporary scene. There is a permanent interplay between both fields: mathematical ideas and methods are applied to logic, and logical ideas and methods find their way into mathematics.

Logic constitutes, of course, an important tool in the domain of philosophy, particularly in philosophy of science, as well as in all areas of (pure or applied) science. And logic did find applications even in technology. It is also intimately linked to natural reasoning and, in general, to all classes of inference. In this chapter, we will consider logic particularly as it relates to valid, deductive inference.

Our main goal in what follows is to give an idea of a new kind of logic, paraconsistent logic, which has been developed, in particular, in Latin America, and which now has applications in a variety of domains. As will become clear shortly, and roughly speaking, paraconsistent logic is a non-classical logic in which not everything follows from contradictions of the form $A$ and not $A$. The development of this logic has had a significant impact on the work in logic and its philosophy in Latin America.

Paraconsistent Logic and Latin America

The formulation and development of paraconsistent logic in Brazil eventually led to what has been called the "Brazilian school of paraconsistency," which combines a pluralism about the many existing paraconsistent logics and an interest in applications of these logics to various areas of research (from the foundations of physics to artificial intelligence) with a lack of commitment to the existence of true contradictions. (We will return to these points below.) This proposal is distinct from other approaches to paraconsistency developed, for instance, in Australia, where a commitment to the existence of true contradictions plays a crucial role (see Priest, 2006a). The impact that paraconsistent logic has had on the theoretical landscape in logic and its philosophy
Thinking about Logic

When one talks about logic, it should be pointed out that, normally, any logic \( L \) is based on a language \( L \), and systematizes the inferences that can be expressed or formulated in that language. The inferences justified by the canons of \( L \) are called \( L \)-valid; the \( L \)-valid inferences are also called \( L \)-deductions. From this point of view, \( L \) is seen as a deductive logic.

However, in \( L \) it may be possible to formulate inferences that are not \( L \)-deductive in accordance with \( L \), but that, nonetheless, deserve to be codified and analyzed. Such patterns of reasoning can be called \( L \)-inductions, and they must display some characteristics that entitle them to be described in this way. For example, the patterns of reasoning should display some degree of plausibility. In this chapter, however, we will discuss only deductive logic, or simply logic. We are mainly interested in deductions as valid inferences, and in related concepts.

The language \( L \) usually contains connectives: \( \rightarrow \) (if . . . then . . . ), \( \neg \) (and), \( \lor \) (or), \( \leftrightarrow \) (if and only if), \( \exists \) (existential quantifier); \( \forall \) (for all); \( \exists \text{there exists} \); as well as other components, such as predicate symbols, variables, and the equality symbol: \( = \). (See, for example, Quine, 1982, and the articles on logic and related topics in Edwards (Ed.), 1972, and Jacquette (Ed.), 2006.)

Of special importance is classical logic. It is one of the most important logics, and the one that is more widely used in the contemporary context. Aristotle, who effectively created logic as a field of study, created what can be called traditional, syllogistic logic. Much later, authors such as C. Boole, G. Frege, B. Russell, and D. Hilbert developed these ideas further, going beyond traditional logic in many ways, and the result is what is now called classical logic. With suitable assumptions, traditional logic can be obtained from classical logic, in the sense that the syllogisms that are considered valid according to traditional logic are considered valid arguments according to classical logic. For this reason, when we talk about classical logic in this chapter, we will consider traditional, syllogistic logic as part of it. (For references and discussion of the history of logic, see Kneale & Kneale, 1988; Edwards (Ed.), 1972; Jacquette (Ed.), 2006.)

From the time of Aristotle to the beginning of the twentieth century, there existed basically one and only one logic: classical logic, which always was the logic. But particularly during the twentieth century, numerous new logics were formulated. Some of them extended classical logic: for instance, classical modal logic (which introduces modal operators of necessity and possibility), classical tense logic (which introduces time operators), and classical deontic logic (which introduces operators modeling what is permissible and obligatory). Other logics, however, change or limit the axioms or the rules governing classical logic. For example, in intuitionistic logic, the principle of excluded middle (namely, \( A \vee \neg A \), where \( A \) is a proposition or sentence) is not valid in general; in many-valued logic, there are more than two truth-values (for instance, true, false, and indeterminate); in quantum logic, the distributive law (i.e., \( P \wedge (Q \lor R) \leftrightarrow (P \wedge Q) \lor (P \wedge R) \), where \( P \), \( Q \), and \( R \) are propositions or sentences) is not generally valid. Logics in the first group, which offer extensions of classical logic, are complementary to the latter. Logics in the second group, which challenge the validity of certain principles of classical logic, are rival to the latter. They can be considered heterodox logics. We will examine below exactly where in this divide paraconsistent logic belongs.
The Nature of Paraconsistent Logic

One of the historically central principles (or laws) of classical logic is the principle of non-contradiction (it is also called the principle of contradiction). The principle has many formulations:

\[ \neg (A \land \neg A), \]

where \( A \) is a sentence and \( \neg \) and \( \neg \) are, as noted above, the symbols for negation and conjunction, respectively. In this formulation, the law of non-contradiction states that it is not the case that a proposition (or sentence) and its negation are both true. So, if \( A \) is true, \( \neg A \) is false, and if \( A \) is false, \( \neg A \) is true. One commonly says that \( A \) and \( \neg \neg A \) cannot be simultaneously true. This is the propositional form of the law.

However, the law also has another formulation: if \( P \) is a one-place predicate, \( \forall \) is a variable, and \( \forall \) is the universal quantifier, then it is true that:

\[ \forall x \neg (P(x) \land \neg P(x)). \]

That is, an object cannot possess and not possess a property. (The traditional logicians would add: "at the same time and from the same point of view," although this addition is not really necessary.) Thus, there is a second, predicate version of the principle of non-contradiction.

Other formulations are still possible. Consider, for example, the following:

\[ \forall x \forall y \neg (Q(x,y) \land \neg Q(x,y)). \]

In this case, \( Q \) is a two-place relation, and the principle states that objects cannot be both related and not related by the relation \( Q \). (As a rough example: all human beings cannot be both married and unmarried.)

The last two versions of the principle -- (2) and (3) -- are the predicate formulations. Version (1), the propositional, in a certain sense implies the predicate forms (2) and (3). So, from now on, when we talk about the principle of non-contradiction, we are making reference to form (1).

One point should be noted, though. Usually, strong logics are stratified in several orders, or levels: first-order logic, second-order logic, ..., high-order logic, where the latter involves all orders. For this reason, there are various forms of the principle of non-contradiction, depending on the order of the logic.

Some terminology is now needed. In classical logic, a formula of the form:

\[ A \land \neg A \]

is called a contradiction; sometimes, the pair \( A \) and \( \neg A \) is also said to be a contradiction. A theory \( T \) is a set of sentences closed by deduction; that is, \( T \) contains all logical consequences of its members (\( T \) is, thus, deductively closed).

Let \( T \) be a theory. \( T \) is trivial if it does contain all sentences of our language \( L \), which, in the case we are discussing now, is the language of classical logic, since at the moment this is the only logic we are considering. (However, the definitions we offer are general, and they cover all logics.) If \( T \) is not a trivial theory, \( T \) is non-trivial. \( T \) is inconsistent if \( T \) has theorems of the form \( A \) and \( \neg A \); otherwise, \( T \) is consistent.

In classical logic, a theory \( T \) is trivial if, and only if, \( T \) is inconsistent. (This happens in most logics.) The reason for this fact is that, in classical logic, from a contradiction one can deduce any formula whatsoever. Moreover, if \( A \) and \( \neg A \) are theorems of the theory \( T \), so is \( A \land \neg A \), and conversely. In synthesis, in classical logic (and in most logics), if a theory \( T \) is inconsistent (or, what means the same, contradictory), then \( T \) is trivial. Thus, it loses its relevance, given that it is unable, for example, to systematize our experience, and it is practically useless. After all, if a set of premises \( \Delta \) is inconsistent, any proposition whatsoever can be deduced from \( \Delta \). Thus, the mere presence of contradictions in one's premises or the employment of contradictory sets of premises constitutes a problem -- in fact a serious problem -- for reasoning.

There are circumstances, however, that motivate one to build a logic that can be the underlying logic of inconsistent but non-trivial theories. The latter are theories that, despite being inconsistent, do not entail everything (that is, they do not have every sentence in the language as a theorem). Clearly, the logic in question cannot be classical logic, or most extant logics, since they are trivialized by a contradiction -- in the sense that every sentence in the language can then be derived. These logics do not allow for inconsistent but non-trivial theories.

We have here, then, a syntactic characterization of paraconsistent logic (that is, roughly, a characterization in terms of the formal features of the logic, independently of issues of truth); a logic is paraconsistent if it can be the basic logic of inconsistent but non-trivial theories. But there is also a semantic formulation of paraconsistent logic (that is, a formulation that is ultimately made in terms of truth), and which is loosely equivalent to the syntactic version. Roughly speaking, if the principle of non-contradiction is not valid in general, then there are true contradictions, and conversely. So, a logic \( L \) is paraconsistent if the law of non-contradiction is not valid in \( L \).

The syntactic and the semantic definitions of paraconsistent logic are informal, and thus, they are not precise. Nonetheless, they are almost equivalent, and any one of these can be taken as a rough delineation of the domain of paraconsistency. It turns out that there are several (in fact, infinitely many) paraconsistent logics, and all of them satisfy the conditions just offered for paraconsistency (see da Costa, Krause, & Bueno, 2007).

There are numerous situations that motivate the convenience and appropriateness of theories that are inconsistent but non-trivial (that is, paraconsistent theories). This occurs, for instance, in set theory, which taken informally and intuitively is inconsistent, due to Russell's paradox. One may think that for every property there is a set of objects corresponding to that property (namely, the objects that have that property). For example, for the property is a tiger, there is a set of objects that have that property: the set of tigers. For the property is a number, there is a set of objects that have that property: the set of numbers. What Russell found out, however, is that there are properties for which there is no corresponding set. Consider the property is a set of sets that are not members of themselves. Let's call such a "set" Russell's set or, more simply, \( R \). Now, consider whether Russell's set is a member of \( R \). Suppose that Russell's set is not a member of \( R \). In this case, given the definition of \( R \), Russell's set is a member of \( R \). Alternatively, suppose that Russell's set is a member of \( R \). In this case, given the definition...
of R, Russell’s set is not a member of R. Thus, Russell’s set is a member of R only if, and only if, it is not a member of R. And from this, it follows that Russell’s set is a member of R and is not a member of R—a contradiction. This shows that our intuitive principle according to which for every property there is a set of objects that have such a property is inconsistent. This, of course, a big surprise, given that this principle, prima facie, seems to be obviously true.

However, if one adopts an underlying paraconsistent logic, this informal, intuitive set theory is transformed into a paraconsistent theory. In this case, there will be a Russell set, which of course is an inconsistent object, but is not a trivial one, in the sense that it has every property. As any object, Russell’s set will have some properties and lack others, and these properties can be studied in a paraconsistent set theory (for details, see da Costa et al., 2007).

There are many other examples in other branches of mathematics, in science, and in philosophy that motivate the need for accommodating inconsistent but non-trivial theories. For example, the early formulation of the calculus in terms of infinitesimals seems to have been inconsistent. An infinitesimal is a positive number (a number that is not zero), but which is smaller than any other number. When the marquis De l’Hospital wrote the first textbook of the calculus, which was published in 1696, his first principle stated that two magnitudes that differ by an infinitesimal are the same. He is here acknowledging that two distinct magnitudes are actually identical— as long as they differ only by an infinitesimal. It is not surprising that much care was needed when infinitesimals were used in the calculus. And it is interesting to note that, despite the inconsistency, Leibniz, Newton, l’Hospital, and others who worked in the early formulation of the calculus managed to obtain the correct results about the theory. They certainly did not derive everything from their inconsistent principles. In fact, this is an interesting example of an inconsistent, and certainly non-trivial, theory. (For a fascinating discussion of the history of the calculus, see Robinson, 1974.)

The same happens with Meinong’s theory of objects, which offers a systematic framework to classify different kinds of objects and their status (Meinong, 1960), and with certain systematizations of dialectics. In both cases, the theories may be inconsistent, but they were certainly non-trivial—not every sentence follows from them. Paraconsistent logic then offers the resources to formulate these theories in a way that preserves some of their central features: their inconsistency, simplicity, and elegance. (The early formulation of the calculus was an elegant theory, and so was Meinong’s.) And this is done without logical chaos; that is, without the triviality that emerges from an inconsistent theory formulated in classical logic.

A History of Paraconsistent Logic

The history of paraconsistent logic is a complex and fascinating affair. With hindsight, even the first systematic formulation of logic in Aristotle’s hands was already paraconsistent. After all, a syllogism with inconsistent premises does not have any arbitrary sentence of the language in use as a valid conclusion. Consider, for instance, this case: (P1) No person is mortal. (P2) Some person is mortal. Therefore, every person is mortal. Clearly, the premises of this argument are inconsistent, but the conclusion does not follow from them according to syllogistic logic. There is a sense of relevance in this logic—a requirement that the premises of a valid argument be relevant to the conclusion—that prevents arguments such as this from being valid. Although Aristotle perhaps would not have put the point this way (given that the concept paraconsistency had not been explicitly formulated at the time), syllogistic theory is indeed paraconsistent.

As part of her account of the history of paraconsistent logic, A. I. Arruda wrote:

Several philosophers since Heraclitus, including Hegel, until Marx, Engels, and the present-day dialectical materialists, have proposed the thesis that contradictions are fundamental for the understanding of reality; in other words, they claim that reality is contradictory, that is to say, that Hegel’s thesis is true of the real world. Hegel’s thesis is the statement that there are true contradictions.

Clearly, if one accepts Hegel’s thesis, one has to employ a new kind of logic (paraconsistent logic), in order to study inconsistent but non-trivial theories. Strangely enough, philosophers who accept Hegel’s thesis have not established any formal system of paraconsistent logic. Instead of this, some of them have proposed the so-called dialectical logic, whose nature is rather obscure.

According to Lukasiewicz [...], Aristotle had already an idea of the possibility of derogation of [that is, the possibility of abandoning] the principle of contradiction, and consequently, the possibility of paraconsistent logic. (Arruda, 1980, p. 6)

It is indeed an interesting feature that from the beginning of the development of logic as a field of inquiry, paraconsistent ideas have been entertained, although they were not, of course, developed at that point. And despite the fact that Aristotle considered the possibility of abandoning the law of non-contradiction in the book 1 of Metaphysics (571a 8–9); see Barnes (2d.), 1984), he ultimately argued against the intelligibility of this move. (Aristotle’s arguments, however, are far from being conclusive; see Priest, 2006b, pp. 7–42.)

Arruda continues:

The first logician to construct a system of paraconsistent logic was S. Jaskowski (in Poland, in 1948 and 1949), following a suggestion of Lukasiewicz. He called his system discursive (discursive) logic. (Arruda, 1980, p. 9)

Arruda also notes:

Jaskowski had already constructed a paraconsistent propositional calculus, but [the Brazilian] N. C. A. da Costa is actually the founder of paraconsistent logic. Independently of the work of Jaskowski, he started in 1958 [...], to develop some ideas which led him to the construction of several systems of paraconsistent logic, including not only the propositional level but also the predicate level (with and without equality), the corresponding calculus of descriptions, as well as some applications to set theory. Da Costa’s systems were extended and studied by several authors [...]. Da Costa and his collaborators investigated also various other systems of paraconsistent logic, some of them having intimate connections with relevant logic. (Roughly speaking, this is a form of paraconsistent logic that requires that the premises of a valid argument be relevant to the conclusion.) [...]. In the last years many logicians contributed to the development of paraconsistent logic (some of them quite independently of the works of Jaskowski and da Costa). (Arruda, 1986, pp. 10–11)
It is important to note that some logicians elaborated certain paraconsistent logical calculi, although they did not conceive the possibility of paraconsistent logic as a new kind of logic. (This is the case, for instance, of some of the work done by D. Nelson in 1959.)

Paraconsistent logic can be viewed as a heterodox logic, which is then a rival to classical logic. From this perspective, both logics are incompatible. However, paraconsistent logic can also be viewed as a complement to classical logic. From this stance, its main concepts and laws are taken to be different from the corresponding ones of classical logic. For example, paraconsistent negation is then distinct from classical negation, obeying proper principles. In general, most heterodox logics can also be conceived as complements to classical logic, since they share several principles. The significance of paraconsistent logic in relation to classical logic depends, thus, on philosophical assumptions about the interpretation of these logics. It is, therefore, an issue in the philosophy of logic.

Many systems of paraconsistent logic have in them classical logic as a kind of “sub-logic.” In particular, when paraconsistent logic and classical logic are used to deal with consistent contexts (that is, those that do not involve any contradiction), they yield the same results, in the sense that exactly the same inferences are sanctioned as valid. Thus, the opposition between classical logic and paraconsistent logic is not as irreconcilable as one may think.

Paraconsistent logic, as logic in general, has numerous applications. (a) In philosophy, paraconsistent logic is used to systematize certain inconsistent theories, such as dialectics and Meinong’s theory of objects, since the latter seems to recognize objects that are neither entities nor non-entities, such as objects that are accessible to our thoughts (Meinong, 1960). Paraconsistent logic also has a role in the philosophical analysis of concepts such as those of negation and implication, and, in philosophy of physics, it shows how to make paraconsistently compatible theories, such as quantum mechanics and general relativity, which are mutually inconsistent. (Of course, the latter application does not hamper the search for a classical unification of these theories.)

(b) In physics and mathematics, paraconsistent logic finds applications in the formalization of inconsistent theories, similar to Bohr’s atomic model and plasma theory, and even the early formulation of the calculus (for a discussion of inconsistent mathematics, see Mortensen, 1995, and da Costa et al., 2007).

(c) Finally, technology offers another domain of application of paraconsistent logic, for example, in traffic control in large cities, in train circulation, in computing, and in artificial intelligence. (A detailed account of paraconsistent logic and topics of paraconsistency in general can be found in da Costa et al., 2007.)

Philosophical Aspects of Paraconsistent Logic

Several issues need to be addressed here. In particular, given that there are infinitely many paraconsistent logics (see da Costa et al., 2007), the issues arise as to whether there is a true paraconsistent logic, and how to choose between such logics. As will become clear, the choice between such logics is ultimately a pragmatic and context-dependent matter, largely dependent on the details of the applications at hand. These points will be developed below.
There is no argument on purely observational grounds that could settle the issue. It is possible, of course, to select one of these logics on pragmatic grounds, but these grounds are certainly not enough to establish a substantive claim about the world. For instance, if one of these logics makes the modeling of the inconsistency in question easier, why should this be taken as a reason for this logic to be true? Simplicity may well be a sensible criterion to adopt on pragmatic grounds, but to claim that a logic selected on this basis is (likely to be) true is to conflate pragmatic and epistemic considerations. Why should the world conform to our cognitive limitations? Of course, it might well do. However, to establish this claim demands an argument that goes beyond the observable: it requires a metaphysical commitment to the simplicity of reality. And, to a certain extent, this is as "strong" as the claim that there are true contradictions.

After all, both make substantial assertions about the world that transcend empirical observation. Both are metaphysical claims. It turns out that an alternative program of interpretation of inconsistencies can be devised in which no commitment to this kind of metaphysics is required. The idea is first to avoid the claim that inconsistent theories are true; they are partially true (or quasi-true) at best. The notion of partial truth can receive a formal treatment (see da Costa & French, 2003). But for the purposes of this chapter, it suffices to note that a sentence S is partially true if it models adequately only part of a given domain D, without offering a complete description of the latter. (In a precise sense, S is consistent with a true description of D.) With regard to inconsistent theories, all that is needed is to determine their partial truth, since one need not be committed (and probably should not be committed) to their full truth.

A formal underpinning to agnosticism with regard to true contradictions can then be provided. By replacing the notion of truth by the weaker notion of partial truth, it is possible to withhold the commitment to "inconsistent objects." After all, these objects are not found in all of the structures under consideration.

Descartes once remarked (in his Principles of Philosophy, iv, 204) that: "with regard to those things that our senses cannot perceive, it suffices to speculate how they can be." The same can be said about inconsistent theories. With the notion of partial truth, it is possible to accommodate this point formally, since partial truth is strictly weaker than truth, and does not commit one to anything beyond the assertion that certain structures are possible, given some paraconsistent logic.

At this point, it is worth revisiting Quine's slogan about ontological commitment, making explicit its dependence on the underlying logic. Depending on the logic that is adopted, different commitments will emerge. In this way, it becomes clear that this slogan is not the only criterion to adjudicate between alternative logics. However, if someone ends up not being committed to an ontology of actual inconsistent physical objects, is that person committed to inconsistent mathematical entities? This depends, of course, on how the relevant mathematical theory is interpreted. Does the inconsistent mathematical theory provide a true description of the mathematical "world"? Again, if the description of this "world" is made at best in partially true terms, no commitment to inconsistent mathematical entities is forthcoming. There is, of course, a whole story to be told here, but for the present purposes, it suffices to note that it is possible to provide an entirely syntactic formulation of paraconsistent set theory, in which various inconsistent theories can be embedded, such that the only
commitment is to a countable language (see da Costa, Bueno, & French, 2005). Thus, in a certain sense, no special commitment to mathematical objects is required either.

Thus, a “package” can be offered to accommodate inconsistencies. It is characterized by (1) the claim that inconsistent theories are partially true at best; (2) an agnosticism with regard to the existence of true contradictions and a nominalism about inconsistent mathematical entities; and (3) a reevaluation of Quine’s view about ontological commitment, emphasizing its dependence on the underlying logic.

The striking feature of this “package” is its logical pluralism, on the one hand, and the fact that it is possible to adopt it to make sense of paraconsistent logic with no commitment to actual “inconsistent objects.” The logical pluralism derives from point (3) above. Depending on the domain under study, different kinds of logic may be appropriate. For instance, if someone intends to model the constructive features in mathematical reasoning, an intuitionistic logic is the best alternative; if someone is concerned with inconsistent bits of information, the use of a paraconsistent logic is the strategic option. In particular, there is no rejection of classical logic here: it has its own domains and applications. To this extent, while dealing with distinct domains, paraconsistent logic and classical logic are complementary rather than rivals. (They become rivals only when applied to the same inconsistent domain. The rivalry derives from the fact that they provide different accounts of the logical connectives.)

But in the application of paraconsistent logic, for example to formulate the theory of the Russell set and other inconsistent objects, there is no need to be committed to the existence of “inconsistent entities” – this is the point of claims (1) and (2) above. The resources of paraconsistent logic can be invoked to draw consequences from inconsistent theories without triviality, but with no commitment to the truth of the theories in question: they can be at best partially true.

In this way, a non-committal (agnostic) interpretation of paraconsistency can be offered. This interpretation uses paraconsistent logic in a way that does not require the existence of “inconsistent objects.” These objects, either mathematical or physical, can be accommodated without requiring an ontology that includes them. In particular, inconsistent mathematical theories can be studied in the context of paraconsistent logic, but it is not necessary to countenance the existence of the entities the theories are taken to be about.

Related chapters: 14 Analytic Philosophy; 26 Philosophy of Science; 31 Deontic Logic and Legal Philosophy; 34 Formal Epistemology and Logic.

References


Further Reading


